**Tata Institute of Fundamental Research**
**Centre for Applicable Mathematics**

# Survey of Differential Inclusion approach in Nonlinear PDEs

**M.Phil Thesis**

by

**Swarnendu Sil**

**Thesis Advisor: Prof. M. Vanninathan**

**Abstract**

In this thesis the framework of compensated compactness and its relation with convex integration and Baire category methods, long familiar techniques in the context of differential inclusion problems has been explored. Compensated compactness theory has originated from and has developed in the context of nonlinear evolution equations of mathematical physics, especially nonlinear elasticity and fluid flow problems where essentially we are dealing with nonlinear hyperbolic systems. Differential inclusion problems and the methodology to resolve differential inclusions originated from questions like optimal control, immersion-theoretic topology, differential geometry, microstructure and phase transition problems in calculus of variations. The basic theme of these two line of thought and the scopes of these two methods looks rather disjoint at first glance. Some recent developements has shown that these two frameworks can be effectively integrated in some cases to produce new results. A survey of the basic ideas underlying these methodologies and how these two apparently disconnected lines of thought collaborate to enable us to produce infinitely many irregular solutions, which are compactly supported in time, to the Euler equations for incompressible fluids is presented.

**Keywords:** compensated compactness, differential inclusion, gradient inclusion, convex integration, Baire category, incompressible Euler equations.

# Contents

# Chapter 1

# Introduction

In this thesis, *oscillations in nonlinear partial differential equations* is, in a sense, our main theme. More particularly, the framework of *compensated compactness*, introduced by Luc Tartar (see [34]), to analyze oscillations in sequences of approximate or exact solutions to nonlinear partial differential equations (henceforth PDEs) and some recent developments along these chain of ideas are our main concern.

Many nonlinear partial differential equations can be written as a *system of linear PDEs* and a *nonlinear pointwise constitutive relations.* The motivation for the idea comes from physics, since most PDEs in physics are indeed derived using this very principle in the first place. One figures out the *constitutive relations* directly from the physical law and one has the *balance equations* for certain quantities. One then substitutes the constitutive relations into the balance equations and obtain the final form of the PDE. Since most equations in physics are derived in this way in the first place, there can be little doubt that these nonlinear PDEs can obviously be cast into that form. What is far from clear and in fact, rather counter-intuitive that casting into this form will any way help to obtain the solution to those nonlinear PDEs. Apparently, we are just backtracking our steps and that should not, in any way help us to stand any better than where we were in the first place. We have not actually achieved anything thus far. We have not got rid of the nonlinearity. We just transferred it from the equation itself to the constitutive relations.

But let us recall the most general type of problem we face while trying to prove existence of solutions for nonlinear equations. Following usual procedure, we want to prove existence first in a sufficiently large space, the advantage being the ease of ensuring existence of a solution and the price paid is the 'weaker' notion of a 'solution'.[1] Using the usual method of truncation, regularization etc and using functional analytic methods, we can obtain a sequence of approximate solutions, which converges only weakly in some suitable function space [2] . But nonlinearity behaves badly with respect to weak convergence and we can not, in general, pass to the limit in the equation to claim that the weak limit of the sequence of approximate solution is itself a weak solution of the equation. One way to get over this obstacle is usual *compactness* arguments. We try to derive enough uniform *apriori bounds* for the derivatives of the sequence of approximate solutions so as to enable us to use *compact embedding* theorems to improve the weak convergence to strong convergence. But it is not always possible to deduce good enough estimates. However it is important to note that although having a uniform bound on the derivatives makes the job easy, it is by no means necessary for strong convergence of the sequence. Another crucial fact is that the lack of compactness, that is, the failure of a weakly convergent sequence to be strongly

[1] Mostly we want to prove existence of a *weak solution*, which does not satisfy the equation in a *classical* sense, but only in a weak sense, possibly in the sense of *distributions* but having some degree of *regularity* in the sense of better *integrability* conditions.

[2] These function spaces are spaces of distributions, generally *Lebesgue* or *Sobolev* type spaces, that is, spaces with *integral type norms.*

convergent stems from oscillations in the sequence.[3] So it might well be a good idea to explore the role the differential operator plays to admit or exclude certain oscillations. But *we want to disentangle the role of the differential structure and the algebraic nonlinear structure of the nonlinear operator.* The specific structure of the nonlinearity suppresses certain oscillations while it makes some other types of oscillation worse. Similarly, the differential structure prohibits some oscillations and allows some other. If the differential structure allows only those oscillations that can be suppressed by the nonlinearity, then we shall have strong convergence, that is, compactness, due to the compensation effect of these two factors. Casting nonlinear PDEs in this form enables us to treat and analyze these two factors separately.

We consider nonlinear PDEs that can be expressed as a system of linear PDEs (*balance Laws*)

$$\sum_{i=1}^{m} A_i \partial_i z = 0 \tag{1.0.1}$$

coupled with a pointwise nonlinear constraint (*constitutive relations*)

$$z(x) \in K \subset \mathbb{R}^d \quad \text{a.e. } x \in \Omega \tag{1.0.2}$$

where $z : \Omega \subset \mathbb{R}^m \to \mathbb{R}^d$ is the unknown state variable.

The general problem of the compensated compactness framework can now be stated as follows: Describe the oscillations in a weakly convergent sequence of functions $z^\epsilon : \Omega \subset \mathbb{R}^m \to \mathbb{R}^d$ subject to linear differential constraints, called *balance laws*, of the form

$$\sum_{i=1}^{m} A_i \partial_i z^\epsilon = \phi^\epsilon \tag{1.0.3}$$

and nonlinear algebraic constraints, called *constitutive relations*, of the form

$$\{z^\epsilon(y)\} \subset M \tag{1.0.4}$$

for almost all $y$ in $\Omega$. Here $A_i$ denotes a constant $s \times d$ matrix with $s$ arbitrary and fixed. $M$ is a subset of the *state space* $\mathbb{R}^d$ and is usually a manifold. Thus the object of study can be regarded as a weakly convergent sequence of solutions or subsolutions to a first order system of differential equations with values in a submanifold of the state space. In compensated compactness theory, one is particularly interested in determining how differential and algebraic constraints collaborate to suppress oscillations in $z^\epsilon$. One of the goals of the theory is to identify those feature of the algebraic structure of the symbol of the differential system (1.0.3) and the constitutive manifold $M$ that admit or exclude oscillation in the sequence $z^\epsilon$.

However, the general problem we discussed is by no means easy. Enough work remains to be done in many directions to able to apply this program successfully to large classes of nonlinear PDEs. One idea is to carry out a *plane wave analysis* , localized in state space, of the balance laws (1.0.1) or (1.0.3) to determine the *wave cone* associated with it. This determines the structure of oscillations permitted by the differential constraints. However, the focus of this thesis is to discuss a relatively new method of obtaining solutions to nonlinear PDEs, which uses all the ideas mentioned above, but in a strikingly different way.

We shall refer to the new method mentioned above as the *the differential inclusion techniques in nonlinear PDE.* This method is not only relatively new, but also encountered some recent successes and is showing enough promise as a fundamentally new and original approach. The crucial idea underlying

[3] Concentration effects can also be responsible, but if only concentration effects are present, we can get almost everywhere convergence.

this method is that here we do not want to ascertain that the differential structure kills all oscillations that the nonlinearity can not suppress. We want something which is quite opposite in a sense. We want the differential structure to allow many types of oscillation, so that the balance laws (1.0.1) has a large, infinite family of oscillatory solutions. We want this family to be so large that in the one hand, it is possible to ensure that there are rich supply of solutions to (1.0.1), which takes values in some *convex hulls* of $K$,[4] on the other hand, we can perturb these solutions of (1.0.1) by adding *controlled oscillation* to generate a sequence in such a way that each term of the sequence still remains a solution of (1.0.1), but the image of each term approaches the constitutive set $K$. Moreover, we want to be able to add controlled oscillations allowed by the differential structure in such a way such that the resulting sequence of solutions to (1.0.1), apriori only weakly convergent, actually converges strongly and the strong limit of this sequence, which evidently also a solution of (1.0.1), takes its values in $K$, that is, satisfies (1.0.2) also. The strong convergence is possible precisely because of our ability to add *controlled oscillation* and is proved either by *convex integration* or *Baire category* arguments, two already familiar techniques in the theory of differential inclusions. This technique has another very interesting feature. If the technique can be applied at all to a nonlinear PDE, it not only solves the PDE, but presents us large number of solutions. So clearly, PDEs like uniformly elliptic equations, where nonuniqueness can not be expected is usually beyond the scope of this method. Also, the solutions produced by this method have very limited regularity. Hence this method has been extensively used recently to prove non-uniqueness or to produce very irregular solutions to a number of nonlinear PDEs.

Though our main aim in this thesis, as already mentioned, is not to determine conditions under which the linear differential constraint and the nonlinear algebraic constraints collaborates to prohibit all possible oscillations, but rather to look for situations where these constraints allow a very large class of oscillations, still as we will see, these questions are in fact intimately related and compensated compactness theory will give us many clues regarding our aim.

The major part of this thesis is a survey of already existing literature. Except possibly for the presentation, no claim of originality is made from the part of the author. As a matter of fact, this thesis draws heavily and freely from the large amount of existing literature in this field. An extensive list, though not entirely exhaustive, can be found in the bibliography. However, despite the presence of some high-quality surveys regarding methods of resolving differential inclusions, we feel that there is a need for a survey which presents the material from the perspective of applications to nonlinear PDEs and the relation and connection of these methods with compensated compactness. This thesis, though rather an introductory survey, hopes to fill this gap.

The rest of the thesis is organized as follows. In chapter 2, we introduce the general setting of compensated compactness theory, the wave cone, oscillation variety, Young measures and their applications to elementary compensated compactness theory and describe their connections with differential inclusion approach. In chapter 3, we look into the differential inclusion approach, presenting the basic framework and many known results provable by these techniques. We also explore the literature on differential inclusions and explore, contrast and compare the two methods, namely the convex integration method and Baire category method. We also study an application of these ideas to generate oscillatory solutions to incompressible Euler equation. In the final chapter, a brief summery concluding remarks are presented. In the appendix, differnt notions of convexity and corresponding convex hulls, which are used throughout has been defined.

[4]Here the geometry of the constitutive set $K$ already plays a role, since we also require some convex hulls of $K$ to be sufficiently large too.

# Chapter 2

# Compensated Compactness

## 2.1 General setting

Let $z_n : \Omega \subset \mathbb{R}^m \to \mathbb{R}^d$ be a sequence of functions such that $z_n \rightharpoonup z$ in $(L^\infty(\Omega))^d$ weak $*$. Also $\{z_n\}$ satisfies a system of linear differential constraints, called *balance laws*, of the form

$$\sum_{i=1}^{m} A_i \partial_i z_n = \phi_n \tag{2.1.1}$$

and nonlinear algebraic constraints, called *constitutive relations*, of the form

$$z_n(x) \in K \tag{2.1.2}$$

for all $n$ for almost all $x$ in $\Omega$. Here $A_i$ denotes a constant $s \times d$ matrix with $s$ arbitrary and fixed. $\Omega$ is a bounded open subset of $\mathbb{R}^m$ and $K$ is a subset of the *state space* $\mathbb{R}^d$. Assume also that $\phi_n \to \phi$ in some suitable topology on a suitable function space.
We want to determine when it is possible to assert that

$$\sum_{i=1}^{m} A_i \partial_i z = \phi \tag{2.1.3}$$

and

$$z(x) \in K \tag{2.1.4}$$

for a.e. $x$ in $\Omega$.

## 2.2 Case without information on derivatives

The first question we can ask is whether we can assert that $z(y) \in K$ for a.e $y$ in $\Omega$ without any information on the derivatives of $z_n$, that is, without (2.1.1). More precisely, we are asking if the informations $z_n \rightharpoonup z$ in $(L^\infty(\Omega))^d$ weak $*$ and $z_n(x) \in K$ for a.e. $x$ in $\Omega$ is enough to imply $z(y) \in K$ for a.e $y$ in $\Omega$. The following theorem answers this question in the negative.

**Theorem 2.2.1.** *(1) Let $z_n : \Omega \to \mathbb{R}^d$ be such that $z_n \rightharpoonup z$ in $(L^\infty(\Omega))^d$ weak $*$ and $z_n(x) \in K$ a.e. Then $z(x) \in \overline{conv}(K)$ a.e (where $\overline{conv}(K)$ is the closed convex hull of $K$).*
*(2) Conversely, let $z \in (L^\infty(\Omega))^d$ and $z(x) \in \overline{conv}(K)$ a.e, then there exists a sequence $\{z_n\}$ such that $z_n \rightharpoonup z$ in $(L^\infty(\Omega))^d$ weak $*$ and $z_n(x) \in K$ a.e. for all $n$.*

To prove this theorem, we first need the following lemma.

**Lemma 2.2.2.** *If $f$ is a convex continuous function on $\mathbb{R}^d$, if $z_n \rightharpoonup z$ in $(L^\infty(\Omega))^d$ weak $*$, and if $f(z_n) \rightharpoonup l$ then $l \geq f(z)$ a.e.*

*Proof.* Since f is convex and continuous, we have $f(y) = \sup_{L \in \mathcal{A}} L(y)$, where $\mathcal{A}$ is a countable collection of affine functions. Therefore $f(z_n) \geq L(z_n)$. Passing to the limit we obtain, $l(x) \geq L(z(x))$ a.e. Hence $l \geq \sup_{L\in\mathcal{A}} L(z) = f(z)$ a.e. . □

We can now easily prove theorem 2.2.1.

*Proof.* (of Theorem 2.2.1) (1) Let $L$ be affine and $L \geq 0$ on $K$. Then $L(z_n) \geq 0$ and hence $L(z) \geq 0$, by lemma 2.2.2. Hence $z(x)$ belongs to any countable intersection of half spaces containing $K$ and thus $z(x) \in \overline{conv}(k)$ a.e.
(2) We first construct characteristic functions $\chi_1^n, \chi_2^n, \ldots \chi_q^n$ of disjoint sets such that $\chi_j^n \rightharpoonup \theta_j$ where the numbers $\theta_j$ satisfy $\theta_j \geq 0$, $\sum \theta_j = 1$, and are given in advance.
Let $B$ be the unit cube in $\mathbb{R}^d$. We make a partition of $B$ into disjoint subsets $\{B_j\}$ of $B$ such that $meas(B_j) = \theta_j$ and let

$$\chi_j = \begin{cases} 1 & \text{in} \quad B_j, \\ 0 & \text{in} \quad \cup_{k\neq j} B_k, \end{cases}$$

extending $\chi_j$ outside $B$ periodically. Then $\chi_j^n(x) = \chi_j(nx) \rightharpoonup \theta_j$. Now let $\omega$ be a step function taking values in $conv(K)$, that is, $\Omega = \cup_{j=1}^r \Omega_j$ and $\omega\mid_{\Omega_j} = k_j$ with $k_j \in conv(K)$. Since $z$ can be approached in the strong $L^p$ topology by a sequence of such step functions, it is enough to show that $\omega$ can be approached in $L^\infty$ weak $*$ by functions taking values only in $K$. Since $k_j = \sum_{l=1}^{q_j} \theta_{jl} k_{jl}$ with $\theta_{jl} \geq 0$, $\sum_l \theta_{jl} = 1$, and $k_{jl} \in K$, we can construct sequences of characteristic functions $\chi_{jl}^n$ by above, such that $\chi_{jl}^n \rightharpoonup \theta_{jl}$ in $L^\infty$ weak $*$. Define $\phi_n$ on $\Omega$ by $\phi_n\mid_{\Omega_j} = \sum_l \chi_{jl} k_{jl}$. Then $\phi_n \rightharpoonup \omega$ in $L^\infty$ weak $*$and $\phi_n$ takes values among $k_{jl}$ and this completes the proof. □

Lemma 2.2.2 immediately suggests the following questions:

Q1. For which functions $f$ do we have $f(z_n) \rightharpoonup f(z)$, that is, $f$ is sequentially weakly continuous ?

Q2. For which functions $f$ do we have if $f(z_n) \rightharpoonup l$, then $l \geq f(z)$, that is, $f$ is sequentially weakly lower semicontinuous?

Q3. If $f(z_n)$ strongly converges to $l$, then is $f(z) = l$ ?

The answers to these questions are collected in the following theorem.

**Theorem 2.2.3.** *Let $z_n : \Omega \to \mathbb{R}^d$ be such that $z_n \rightharpoonup z$ in $(L^\infty(\Omega))^d$ weak $*$ and $z_n(x) \in K$ a.e..*
*Then we have,*

*(1) $z(x) \in K$ if and only if $K$ is closed and convex.*

*(2) $f(z_n) \rightharpoonup f(z)$ in $(L^\infty)^N$ weak $*$ if and only if $f$ is affine on $\overline{conv}(K)$.*

*(3) If $f(z_n) \rightharpoonup l$ in $(L^\infty)^N$ weak $*$, then $l \geq f(z)$ if and only if $f$ is convex on $\overline{conv}(K)$.*

*(4) If $f(z_n) \to l$ (strongly), then $l = f(z)$, if and only if for all $a \in \mathbb{R}^N$ (where $f : \mathbb{R}^d \to \mathbb{R}^N$ is continuous), $f$ is constant on $\overline{conv}(f^{-1}(a) \cap \overline{K})$.*

**Remark 2.2.1.** Note that theorem 2.2.3 is still valid if we replace weak $*$ in $L^\infty$ by weak convergence in $L^p$, $(1 \leq p < \infty)$ with appropriate growth restrictions on $f$. (see [5] )

Before proving the theorem, we need to introduce the important concept of *Young measures* or *parametrized measures.* So we postpone the proof of theorem 2.2.3 for now.

**Definition 2.2.4** (Young measure:)**.** A parametrized measure or Young measure is a family of probability measures, $\nu = \{\nu_x\}_{x\in\Omega}$ , associated with a sequence of functions $u_j : \Omega \subset \mathbb{R}^m \to \mathbb{R}^d$ such that $supp\{\nu_x\} \subset \mathbb{R}^d$ and they depend measurably on $x \in \Omega$, which means that for any continuous function $f : \mathbb{R}^d \to \mathbb{R}$, the function

$$\bar{f}(x) = \int_{\mathbb{R}^d} f(\lambda)\mathrm{d}\nu_x(\lambda) = \langle f, \nu_x\rangle$$

is measurable. The fundamental property of parametrized measures of this family of probability measures is that, whenever $\{f(u_j)\}$ converges weakly $*$ in $L^\infty(\Omega)$ (or more generally weak in some $L^p(\Omega)$), the weak limit can be identified with the function $\bar{f}$.

Since Young measure characterizes all weak limits of all function $f(u_j)$ for all functions $f$, the knowledge of Young measure of a sequence $\{u_j\}$ is a useful tool to answer questions like the ones raised above.

Now we prove the existence for such a family of probability measures. The following two theorems will give ensure us the existence of Young measure of a sequence (upto a subsequence) under sufficiently general conditions.

**Theorem 2.2.5.** *Suppose $K$ is bounded in $\mathbb{R}^d$ and $\Omega$ is an open set in $\mathbb{R}^m$. Let $u_n : \Omega \to \mathbb{R}^d$ be a sequence of measurable functions such that $u_n(x) \in K$ a.e. Then there exists a subsequence $u_j$ and a family of probability measures $\{\nu_x\}_{x\in\Omega}$ on $\mathbb{R}^d$ (depending measurably on $x$) with $supp(\nu_x) \subset \overline{K}$ such that if $f$ is a continuous function on $\mathbb{R}^d$ and*

$$\bar{f} = \langle \nu_x, f(\lambda)\rangle \ a.e.$$

*then,*

$$f(u_j) \rightharpoonup \bar{f} \ in\ L^\infty(\Omega)\ weak * .$$

**Theorem 2.2.6.** *Let $\Omega \subset \mathbb{R}^m$ be a measurable set and let $u_n : \Omega \to \mathbb{R}^d$ be a sequence of measurable functions such that*

$$\sup_n \int_\Omega g(|u_n|)\mathrm{d}x < \infty$$

*where $g : [0,\infty) \to [0,\infty)$ is a continuous, nondecreasing function such that $\lim_{t\to\infty} g(t) = \infty$. Then there exists a subsequence $\{u_j\}$, and a family of probability measures $\{\nu_x\}_{x\in\Omega}$ such that whenever the sequence $\{\psi(x, u_j(x))\}$ is weakly convergent in $L^1(\Omega)$ for any Carathéodory function $\psi(x,\lambda) : \Omega \times \mathbb{R}^d \to \mathbb{R}^*$, the weak limit is the function*

$$\bar{\psi}(x) = \int_{\mathbb{R}^d} \psi(x,\lambda)\mathrm{d}\nu_x(\lambda).$$

By taking $g(t) = t^p$ for $1 \le p < \infty$ in theorem 2.2.6, we obtain the important corollary,

**Corollary 2.2.1.** Every bounded sequence in $L^p(\Omega)$, $1 \le p < \infty$, contains a subsequence that generates a parametrized measure in the sense of theorem 2.2.6.

**Remark 2.2.2.** (1) Let $u_n \rightharpoonup u$ in $L^\infty(\Omega)$ weak $*$. Then $u_n \to u$ strongly (in every $L^p(\Omega)$, $1 \le p < \infty$) if and only if

$$\nu_x = \delta_{u(x)} \text{ a.e.}$$

where $\delta_y$ denotes the Dirac measure at $y$.
(2) Observe also that though Young measures captures the oscillation of a sequence , it completely misses any concentration effect that might be present.

Proof of theorem (2.2.5) and the remark above can be found in [34]. The theorem (2.2.6) is proved in [29].

Now we can return to the proof of theorem 2.2.3.

*Proof.* (of Theorem 2.2.3)
(1) is just theorem 2.2.1.
(2) follows immediately from (3), by applying (3) to $f$ and $-f$.
(3) In theorem 2.2.1, we have seen how to construct from characteristic functions a sequence

$$z_n = \sum_{j=1}^{q} \chi_{nj} k_j \rightharpoonup z$$

where $k_j \in K$, and such that $\chi_{nj} \rightharpoonup \theta_j$ with $\theta_j \geq 0$, $\sum_{j=1}^{q} \theta_j = 1$. Then

$$f(z_n) \rightharpoonup \sum_{j=1}^{q} \theta_j f(k_j) = l \geq f(z) = f(\sum_{j=1}^{q} \theta_j k_j)$$

implies that $f$ is convex (and continuous) on $\overline{conv}(K)$. The converse is obvious.
(4) We have the following equivalences,

$$\begin{aligned} f(z_n) \to l \text{ strongly} \quad &\Leftrightarrow \quad \text{image of } \nu_x \text{ by } f \text{ is a Dirac measure } \delta_{l(x)} \\ &\Leftrightarrow \quad supp(\nu_x) \subset f^{-1}(l(x)) \cap \overline{K} \\ &\Leftrightarrow \quad \text{f is constant on } \overline{conv}(supp(\nu_x)) \\ &\Leftrightarrow \quad f(z) = \text{constant} = \langle \nu_x, f(\lambda) \rangle = l(x). \end{aligned}$$

□

We now turn our attention towards somewhat more complicated problems, the case where we do have some information on the derivatives, that is, the case where differential constraints are also present.

## *2.3 Case with information on derivatives*

In the last section, what we basically analyzed is how weak convergence behaves with respect to nonlinearity. To our dismay, we found that weak convergence behaves so badly with respect to nonlinearity that all weakly continuous functions are affine (Cf. theorem 2.2.3, part (2)), that is , *there are no non-linear function which are weakly continuous.* However, all is not lost. With our knowledge of compact embedding theorems (we need $\Omega$ to be bounded with not too irregular boundary in that case), we know that if the weakly convergent sequence satisfies, in addition, an uniform bound for all its partial derivatives, then the sequence (upto a subsequence) converges strongly and hence, in that case, all nonlinear functions are weakly continuous. So we can legitimately hope that there are cases which are intermediate to these two extremes, in a sense that with some information on some derivatives may indeed yield some nonlinear weakly continuous functions. This hope will indeed come out to be true and that is, in essence, is the content of the present section.
We state the problem in the same way as before.$\Omega$ and $K$ are both assumed bounded.
*Hypotheses:*

$$u_1^\epsilon, \ldots, u_d^\epsilon \rightharpoonup u_1, \ldots, u_d \text{ in } L^\infty(\Omega) \text{ weak } * \tag{2.3.1}$$

$$u^\epsilon \in K \text{ a.e.} \tag{2.3.2}$$

$$\sum_{j,k} a_{ijk} \frac{\partial u_j^\epsilon}{\partial x_k} \in \text{ bounded set in } L^\infty \tag{2.3.3}$$

for $i = 1, \ldots, q$ where the $a_{ijk}$ are real constants and where the derivatives $\frac{\partial u_j^\epsilon}{\partial x_k}$ are in the sense of distributions.
The questions are the same as before, that is,

Q1. Does $u(x) \in K$ a.e.?

Q2. For which functions $f$ do we have $f(u^\epsilon) \rightharpoonup f(u)$, that is, $f$ is sequentially weakly continuous ?

Q3. For which functions $f$ do we have if $f(u^\epsilon) \rightharpoonup l$, then $l \geq f(u)$, that is, $f$ is sequentially weakly lower semicontinuous?

Q4. If $f(u^\epsilon)$ strongly converges to $l$, then is $f(u) = l$ ?

**Remark 2.3.1.** A more general question whose solution will answer all the proceeding ones is:

(1) Find the smallest set $\tilde{K}$ such that (2.3.1) and (2.3.2) imply $u(x) \in \tilde{K}$ a.e. Then (Q1) means: is $\tilde{K} = K$?

(2) (Q2) can be interpreted as follows:
Define new function $u^\epsilon_{d+1} = f(u^\epsilon_1, \ldots, u^\epsilon_d)$ and let $K_1 = \{(k, f(k)), k \in K\}$. (Q2) means: is $\tilde{K_1} \subset \{(k, f(k)) : k \in \overline{conv}(K)\}$ ?

(3) (Q3) similarly means: is $\tilde{K_1} \subset \{(k, \lambda) : k \in \overline{conv}(K), \lambda \geq f(k)\}$ ?

The partial answer to these questions will rely heavily on the following definitions.

**Definition 2.3.1** (Oscillation Variety:)**.** The oscillation variety corresponding to (2.3.3) is the set $\mathcal{V}$, defined by,

$$\mathcal{V} := \left\{ (\lambda, \xi) \in \mathbb{R}^d \times \mathbb{R}^m \backslash \{0\} : \sum_{j,k} a_{ijk} \lambda_j \xi_k = 0 \quad for\ i = 1, \ldots, q \right\}. \tag{2.3.4}$$

and

**Definition 2.3.2** (Wave Cone:)**.** The wave cone corresponding to (2.3.3) is the projection of $\mathcal{V}$ on the physical space, defined by,

$$\Lambda := \left\{ \lambda \in \mathbb{R}^d : \exists \xi \in \mathbb{R}^m \backslash \{0\} \quad with \ \sum_{j,k} a_{ijk} \lambda_j \xi_k = 0 \quad for\ i = 1, \ldots, q \right\}. \tag{2.3.5}$$

**Remark 2.3.2.** (1) In the case without derivatives, we have $\mathcal{V} = \mathbb{R}^d \times \mathbb{R}^m \backslash \{0\}$ and $\Lambda = \mathbb{R}^d$. If $\Lambda = \mathbb{R}^d$, this means (2.3.3) contains very little information.

(2) In the compactness case, we have $\Lambda = \{0\}$, and this happens if the list (2.3.3) contains all the derivatives $\dfrac{\partial u^\epsilon_j}{\partial x_k}$, which are thus all bounded.

We now present some necessary conditions.

**Theorem 2.3.3.** *If (2.3.1), (2.3.2) and (2.3.3) imply $u(x) \in K$ a.e., then $K$ must satisfy the following conditions:*

*NC0: $K$ is closed.*

*NC1: If $a, b \in \overline{K} = K$ and $b - a \in \Lambda$, then the segment $[a, b]$ is in $K$.*

*Proof.* NC0 is clear.

NC1: Let $a, b \in K$ and $b - a \in \Lambda$. Then by the definition of $\Lambda$, there exists $\xi \in \mathbb{R}^m$, $\xi \neq 0$, such that

$$\sum_{j,k} a_{ijk} (b - a)_j \xi_k = 0 \quad for \quad i = 1, \ldots, q.$$

For any sequence of functions $f^\epsilon$ we construct

$$u^\epsilon = a + (b-a)f^\epsilon(\xi \cdot x)$$

and we claim that,

$$\sum_{j,k} a_{ijk} \frac{\partial u_j^\epsilon}{\partial x_k} = 0 \tag{2.3.6}$$

holds.
(1) If $f^\epsilon$ is smooth, then

$$\begin{aligned}\frac{\partial u_j^\epsilon}{\partial x_k} &= (b-a)_j \frac{\partial}{\partial x_k}(f^\epsilon(\xi \cdot x)) \\ &= (f^\epsilon)'(\xi \cdot x)(b-a)_j \xi_k.\end{aligned}$$

Therefore,

$$\frac{\partial u_j^\epsilon}{\partial x_k} = (f^\epsilon)'(\xi \cdot x) \sum_{j,k} a_{ijk}(b-a)_j \xi_k = 0$$

and (2.3.6) is satisfied.
(2) By approximation, (2.3.6) is true for all $f^\epsilon$.
Now we make a particular choice of $f^\epsilon$, namely a sequence of characteristic functions such that

$$f^\epsilon \rightharpoonup \theta \in [0,1].$$

Then $u^\epsilon$ takes only the values $a$ and $b$, and so $u^\epsilon \in K$ *a.e.*.
Furthermore

$$u^\epsilon \rightharpoonup (1-\theta)a + \theta b \qquad \text{in} \quad L^\infty \text{ weak } *,$$

so that

$$(1-\theta)a + \theta b \ \in K.$$

Thus,

$$[a,b] \subset K.$$

□

**Corollary 2.3.1.** A necessary condition on $K$ and $f$ so that (Q2) can be answered positively is: If $a_1, a_2, \ldots, a_d \in K,\ \xi \neq 0$ with $(a_j - a_k, \xi) \in \mathcal{V}$ for all $j,k$, then $f$ is affine on $conv\{a_1, \ldots, a_d\}$.

*Proof.* Choose functions $\{u^\epsilon\}$ of $\xi \cdot x$, taking only values $a_1, \ldots, a_n$ and such that

$$f(u^\epsilon) \rightharpoonup \sum_{i=1}^n \theta_i f(a_i), \qquad \theta_i \geq 0, \quad \sum_{i=1}^n \theta_i = 1.$$

Since we require that $f\left(\sum_{i=1}^n \theta_i a_i\right) = \sum_{i=1}^n \theta_i f(a_i)$, it follows that $f$ is affine on $conv\{a_1, \ldots, a_n\}$. □

**Corollary 2.3.2.** A necessary condition on $K$ and $f$ such that (Q3) can be answered positively is: If $a_1, a_2, \ldots, a_d \in K,\ \xi \neq 0$ with $(a_j - a_k, \xi) \in \mathcal{V}$ for all $j,k$, then $f$ is convex on $conv\{a_1, \ldots, a_d\}$.

*Proof.* This is similar to that of Corollary 2.3.1. □

**Remark 2.3.3.** In the case $K = \mathbb{R}^d$, (2.3.2) implies that $f$ is $\Lambda$-convex, that is, convex in any direction belonging to $\Lambda$, i.e for all $\lambda_0 \in \Lambda,\ a \in \mathbb{R}^d,\ f(a+t\lambda_0)$ is convex in $t$. In particular, if $f$ is of class $C^2$, we have,

$$f''(a)(\lambda,\lambda) \geq 0 \text{ for all } a \in \mathbb{R}^d,\ \lambda \in \Lambda$$

We now proceed to look for some necessary and sufficient conditions. However, up to now the question of sufficient conditions has not received a fully satisfactory answer and many problems are still open. But we can give a fairly complete answer when the nonlinearity is quadratic.

**Theorem 2.3.4.** *Suppose we have the following hypotheses*

$(H1')$ $u_i^\epsilon \rightharpoonup u_i$ *in* $L^2(\Omega)$ *weak for* $i = 1, \dots, d$

$(H3')$ $\displaystyle\sum_{j,k} a_{ijk} \frac{\partial u_j^\epsilon}{\partial x_k} \in$ *a compact set (for the strong topology) of* $H^{-1}_{loc}(\Omega)$, *for* $i = 1, \dots, d$.

*Then if $Q$ is quadratic and satisfies $Q(\lambda) \geq 0$ for all $\lambda \in \Lambda$, and if $Q(u^\epsilon) \rightharpoonup l$ in the sense of distributions ( $l$ may be a measure), then*

$$l \geq Q(u) \quad \textit{(in the sense of measures).}$$

**Remark 2.3.4.** (1) $H^{-1}(\Omega)$ is the dual of $H_0^1(\Omega)$.

(2) If $Q$ is quadratic to say that $Q$ is $\Lambda$-convex is equivalent to saying $Q(\lambda) \geq 0$, for all $\lambda \in \Lambda$.

(3) This theorem asserts that if $f$ is quadratic then the necessary condition stated in Corollary 2.3.2 is also sufficient.

*Proof.* 1st step: We perform a translation

$$v_i^\epsilon = u_i^\epsilon - u_i.$$

Then $(H1')$ and $(H3')$ imply that

(1) $v_i^\epsilon \rightharpoonup 0$ in $L^2(\Omega)$ weakly

(2) $\displaystyle\sum_{j,k} a_{ijk} \frac{\partial v_j^\epsilon}{\partial x_k} \in$ a compact set of $H^{-1}_{\text{loc}}(\Omega)$, for $i = 1, \dots, d$

Since $Q$ is quadratic, there exists a bilinear form $q(a,b)$ such that

$$Q(a) = q(a,a).$$

Therefore

$$Q(v^\epsilon) = Q(u^\epsilon - u) = Q(u^\epsilon) - 2q(u^\epsilon, u) + Q(u).$$

But

$$Q(u^\epsilon) \rightharpoonup l \quad \text{and} \quad q(u^\epsilon, u) \rightharpoonup q(u,u),$$

the second statement holds since $q(a,b)$ is linear in $a$ for fixed $b$.
Therefore since $q(u,u) = Q(u)$

$$Q(v^\epsilon) \rightharpoonup l - Q(u) = m.$$

So the theorem is equivalent to showing $m \geq 0$.
2nd step: Next we perform a localization as follows. We let

$$w^\epsilon = \phi v^\epsilon \quad \text{with} \quad \phi \in \mathcal{D}(\Omega)$$

Then

$$supp\ w^\epsilon \subset \text{ a compact set of } \mathbb{R}^m. \tag{2.3.7}$$

$$w^\epsilon \rightharpoonup 0 \text{ in } L^2(\Omega) \text{ weak.} \tag{2.3.8}$$

Also

$$\sum_{j,k} a_{ijk} \frac{\partial w_j^\epsilon}{\partial x_k} = \phi \sum_{j,k} a_{ijk} \frac{\partial v_j^\epsilon}{\partial x_k} + \sum_{j,k} a_{ijk} v_j^\epsilon \frac{\partial \phi}{\partial x_k}.$$

The first term belongs to a compact set of $H^{-1}(\Omega)$ and the second is bounded in $L^2(\Omega)$, and hence belongs to a compact set of $H^{-1}(\Omega)$ (since the inclusion map is compact). Therefore,

$$\sum_{j,k} a_{ijk} \frac{\partial w_j^\epsilon}{\partial x_k} \in \text{a compact set of } H^{-1}(\Omega).$$

Hence extracting possibly a subsequence we have

$$\sum_{j,k} a_{ijk} \frac{\partial w_j^\epsilon}{\partial x_k} \to 0 \text{ in } H^{-1} \text{ strong} \qquad \text{and} \tag{2.3.9}$$

$$Q(w^\epsilon) \rightharpoonup \phi^2 m. \tag{2.3.10}$$

<u>3rd step:</u> To prove $m \geq 0$ it is enough to show that

$$\varliminf_{\epsilon \to 0} \int Q(w^\epsilon) \mathrm{d}x \geq 0 \tag{2.3.11}$$

since this will prove

$$\varliminf_{\epsilon \to 0} \int Q(\phi v^\epsilon) \mathrm{d}x = \langle m, \phi^2 \rangle \geq 0$$

for all $\phi \in \mathcal{D}(\Omega)$, implying $m \geq 0$.
<u>4th step:</u> We use Fourier transform and Plancherel formula. Let us define

$$\widehat{w_j^\epsilon} = \mathcal{F}(w_j^\epsilon) = \int_{\mathbb{R}^m} w_j^\epsilon(x) e^{2\pi i (\xi \cdot x)} \mathrm{d}x \quad . \tag{2.3.12}$$

The Plancherel formula gives

$$\int_{\mathbb{R}^m} v(x) \overline{w(x)} \mathrm{d}x = \int_{\mathbb{R}^m} \widehat{v}(\xi) \overline{\widehat{w}(\xi)} \mathrm{d}\xi \tag{2.3.13}$$

where $v$ and $w \in L^2(\mathbb{R}^m)$ are complex valued.

We extend $Q$ from $\mathbb{R}^d$ to $\mathbb{C}^d$ into an Hermitian form. Recall that

$$Q(\lambda) = \sum_{j,k} q_{jk} \lambda_j \lambda_k$$

with $q_{jk} = q_{kj}$ real. We define

$$\tilde{Q}(\lambda) = \sum_{j,k} q_{jk} \lambda_j \overline{\lambda}_k$$

Hence we have

$$Re(\tilde{Q}(\lambda)) \geq 0 \qquad \text{if } \lambda \in \Lambda + i\Lambda, \tag{2.3.14}$$

since if $\lambda = \lambda_1 + i\lambda_2$ with $\lambda_1, \lambda_2 \in \Lambda$ then

$$\tilde{Q}(\lambda) = (Q(\lambda_1) + Q(\lambda_2)) + i(q(\lambda_1, \lambda_2) + q(\lambda_2, \lambda_1))$$

(where $q$ was defined by $Q(a) = q(a, a)$) and therefore

$$Re\ \tilde{Q}(\lambda) = Q(\lambda_1) + Q(\lambda_2) \geq 0.$$

By the Plancherel formula,

$$\int_{\mathbb{R}^m} Q(w^\epsilon)\mathrm{d}x = \int_{\mathbb{R}^m} \tilde{Q}(\widehat{w}^\epsilon)(\xi)\mathrm{d}\xi = \int_{\mathbb{R}^m} Re\ \tilde{Q}(\widehat{w}^\epsilon)(\xi)\mathrm{d}\xi$$

So (2.3.11) is equivalent to

$$\varliminf_{\epsilon\to 0} \int_{\mathbb{R}^m} Re\ \tilde{Q}(\widehat{w}^\epsilon)\mathrm{d}\xi \geq 0. \tag{2.3.15}$$

But, since *supp* $w_j^\epsilon \subset$ a fixed compact set $\mathsf{C}$ of $\mathbb{R}^m$, we have

$$\widehat{w_j^\epsilon}(\xi) = \int_{\mathsf{C}} w_j^\epsilon(x) e^{2\pi i(\xi\cdot x)}\mathrm{d}x.$$

But $e^{2\pi i(\xi\cdot x)} \in L^2(\mathsf{C})$ and since $w_j^\epsilon \rightharpoonup 0$ in $L^2(\mathbb{R}^m)$ we deduce

$$\widehat{w_j^\epsilon}(\xi) \to 0 \ \text{ for all } \xi, \quad \text{ and } \ |\widehat{w_j^\epsilon}(\xi)| \leq M. \tag{2.3.16}$$

Therefore

$$\widehat{w_j^\epsilon}(\xi) \to 0 \text{ locally ( in } L^2_{\mathrm{loc}}(\mathbb{R}^m) \text{ strong ).}$$

Hence

$$\int_{|\xi|\leq r} \tilde{Q}(\widehat{w}^\epsilon)\mathrm{d}\xi \to 0 \ \text{ as } \ \epsilon \to 0. \tag{2.3.17}$$

Taking the Fourier transform of (2.3.9) leads to:

$$\frac{1}{1+|\xi|}\sum_{j,k} a_{ijk}\widehat{w_j^\epsilon}(\xi)\xi_k \to 0 \quad \text{in } L^2(\mathbb{R}^m) \text{ strong for } i = 1, \dots, q \tag{2.3.18}$$

To finish the proof, we need the following lemma.

**Lemma 2.3.5.** *Suppose that $Q(\lambda) \geq 0$ for all $\lambda \in \Lambda$. Then for all $\alpha > 0$ there exists a constant $C_\alpha$ such that*

$$Re\ \tilde{Q}(\lambda) \geq -\alpha|\lambda|^2 - C_\alpha \left( \sum_{i=1}^{q} \left| \sum_{j,k} a_{ijk}\lambda_j\eta_k \right|^2 \right) \tag{2.3.19}$$

*for all $\lambda \in \mathbb{C}^d$, and for all $\eta \in \mathbb{R}^m$ with $|\eta| = 1$.*

<u>End of proof of Theorem 2.3.4:</u> We take

$$\lambda = \widehat{w^\epsilon}(\xi) \text{ and } \eta = \frac{\xi}{|\xi|}.$$

It follows from 2.3.18 that

$$\frac{1}{|\xi|}\sum_{j,k} a_{ijk}\widehat{w_j^\epsilon}\xi_k \to 0 \text{ in } L^2(|\xi| \geq 1) \text{ strong.} \tag{2.3.20}$$

Using the lemma, we have

$$Re\ \tilde{Q}(\widehat{w^\epsilon}(\xi)) \geq -\alpha|\widehat{w^\epsilon}(\xi)|^2 - C_\alpha \left( \sum_{i=1}^{q} \left| \sum_{j,k} a_{ijk} \frac{\widehat{w_j^\epsilon}(\xi)\xi_k}{|\xi|} \right|^2 \right)$$

Integrating, we get

$$\int_{|\xi|\geq 1} Re\ \tilde{Q}(\widehat{w^\epsilon}(\xi))\mathrm{d}\xi \geq -\alpha \int_{|\xi|\geq 1} |\widehat{w^\epsilon}(\xi)|^2 \mathrm{d}\xi - C_\alpha \int_{|\xi|\geq 1} \sum_{i=1}^{q} \left| \sum_{j,k} a_{ijk} \frac{\widehat{w_j^\epsilon}(\xi)\xi_k}{|\xi|} \right|^2 \mathrm{d}\xi$$

But by (2.3.16) and (2.3.20) we deduce

$$\varliminf_{\epsilon\to 0} \int_{|\xi|\geq 1} Re\ \tilde{Q}(\widehat{w^\epsilon}(\xi))\mathrm{d}\xi \geq M\alpha$$

and this is true for all $\alpha$ as small as desired. Hence,

$$\varliminf_{\epsilon\to 0} \int_{|\xi|\geq 1} Re\ \tilde{Q}(\widehat{w^\epsilon}(\xi))\mathrm{d}\xi \geq 0 \tag{2.3.21}$$

Then (2.3.17) and (2.3.21) yields the result. □

*Proof of Lemma 2.3.5.* We proceed by contradiction. Suppose there exists an $\alpha_0 < 0$ such that for all $C_\alpha = n$ there exists $\lambda^n \in \mathbb{C}^d$ with $|\lambda^n| = 1$ and $\eta^n \in \mathbb{R}^m$ with $|\eta^n| = 1$ such that

$$Re\ \tilde{Q}(\lambda^n) \geq -\alpha|\lambda^n|^2 - n \left( \sum_{i=1}^{q} \left| \sum_{j,k} a_{ijk} \lambda_j^n \eta_k^n \right|^2 \right) . \tag{2.3.22}$$

Extract convergent subsequences such that

$$\lambda^n \to \lambda^\infty \quad \text{and} \quad \eta^n \to \eta^\infty .$$

Then

$$\sum_{i=1}^{q} \left| \sum_{j,k} a_{ijk} \lambda_j^n \eta_k^n \right|^2 \leq \frac{C}{n}$$

where $C$ is a constant. Hence, passing to the limit, we deduce that

$$\sum_{j,k} a_{ijk} \lambda_j^\infty \eta_k^\infty = 0 .$$

Therefore

$$\lambda^\infty \in \Lambda + i\Lambda$$

and hence

$$Re\ \tilde{Q}(\lambda^\infty) \geq 0 .$$

But from (2.3.22) we have also

$$Re\ \tilde{Q}(\lambda^\infty) = \lim_{n\to\infty} Re\ \tilde{Q}(\lambda^n) \leq -\alpha_0 < 0 .$$

which is a contradiction. □

**Corollary 2.3.3.** If $Q$ is quadratic and satisfies $Q(\lambda) = 0$ for all $\lambda \in \Lambda$, and if $\{u^\epsilon\}$ satisfies $(H1')$ and $(H3')$, then $Q(u^\epsilon) \rightharpoonup Q(u)$ in the sense of distributions.

*Proof.* Extract a subsequence such that

$$Q(u^\epsilon) \rightharpoonup l \quad ( \, l \text{ may apriori be a measure } ).$$

Applying the theorem to $Q$ and $-Q$, we obtain $l = Q(u)$, and since this is true for all subsequences the corollary follows. □

**Corollary 2.3.4.** Div-Curl Lemma:

Let

$$v^\epsilon \rightharpoonup v \qquad \text{in } (L^2(\Omega))^m \text{ weak },\tag{2.3.23}$$
$$w^\epsilon \rightharpoonup w \qquad \text{in } (L^2(\Omega))^m \text{ weak },\tag{2.3.24}$$
$$div\ v^\epsilon \to div\ w \qquad \text{in } H^{-1}(\Omega) \text{ strong },\tag{2.3.25}$$
$$curl\ w^\epsilon \to curl\ w \qquad \text{in } (H^{-1}(\Omega))^{m^2} \text{ strong}.\tag{2.3.26}$$

Then

$$v^\epsilon \cdot w^\epsilon \rightharpoonup v \cdot w \qquad \text{in } \mathcal{D}'(\Omega)\tag{2.3.27}$$

and this is the only nonlinear functional that is (sequentially) weakly continuous.

*Proof.* Define $u^\epsilon = (v^\epsilon, w^\epsilon)$ and $u = (v, w)$ . Then $u^\epsilon \rightharpoonup u$ in $(L^2(\Omega))^{2m}$ and the theorem can be applied. The corresponding oscillation variety $\mathcal{V}$ is,

$$\begin{aligned}\mathcal{V} &= \left\{((\lambda,\mu),\xi) : \xi \neq 0,\ \sum_{i=1}^{m} \lambda_i \xi_i = 0 \text{ and } \mu_i\xi_j - \mu_j\xi_i = 0 \text{ for all } i,j\right\}\\ &= \{((\lambda,\mu),\xi) : \lambda \perp \xi \text{ and } \mu \parallel \xi\}\,.\end{aligned}\tag{2.3.28}$$

Thus

$$\Lambda = \{(\lambda,\mu) : \lambda \perp \mu\}\,.\tag{2.3.29}$$

Since $\lambda \perp \mu \Leftrightarrow \lambda \cdot \mu = 0$, hence the quadratic form $Q(\lambda, \mu) = \lambda \cdot \mu$ is identically 0 on $\Lambda$ and this is the only quadratic form with this property. Therefore, by Corollary 2.3.3, the Corollary follows. □

## *2.4 Overview*

Theorem 2.3.4 and its corollaries are certainly satisfying results. Especially the div-curl lemma (Corollary 2.3.4) has been an important result, used widely in the context of nonlinear PDEs and homogenization, variational problems etc. But if the nonlinearity is not quadratic, no necessary and sufficient conditions for weak lower semicontinuity of this type is known till date. In fact, good enough sufficient conditions for weak continuity or weak lower semicontinuity are also not known. However, a number of necessary conditions, other than those presented here and a few sufficient conditions along with applications in nonlinear elasticity, calculus of variations and hyperbolic conservation laws can be found in [34],[35]. The compensated compactness theory has been widely employed in the theory of scalar and system of hyperbolic conservation laws and extensive list of literature exists along these directions (see e.g. [14], [23], [4] etc ).

A step in characterizing conditions under which the weak convergence takes place in the trilinear setting, in one space dimension can be found in [18]. The interaction of three oscillatory sequences can be quite complicated and as can be seen in that paper, even in one space dimension, the phenomenon of resonance plays a role. In higher space dimensions, the geometry of such possible interactions gets very complicated.

So in conclusion, though compensated compactness theory has already produced some useful results, more general viewpoint is needed. A suitable object, like young measures, which can handle all types of nonlinearity and unlike young measures, can use the information encoded in the differential constraints seems necessary for this task. Improvements along these lines can be found in [36] and references therein.

But as we mentioned earlier, our goal is to understand a more recent and in a sense unexpected developments of this framework of the present chapter to certain classes of nonlinear problems. With this in mind, we now turn towards reviewing the material presented so far in this chapter in a new light.

## *2.5 Perspectives*

Recall the general setting discussed in Section 2.1. There we asked the question whether given a sequence of approximate solutions $\{z_n\}$ satisfying (2.1.1) and (2.1.2), when is it possible to assert that the weak limit $z$ of the sequence $\{z_n\}$ satisfies (2.1.3) and (2.1.4). But this question was only the means to an end. What was the end we actually desired? We wanted to find a solution $z$ of (2.1.3) and (2.1.4). Keeping this in mind, we now ask a different question:
Given a sequence of approximate solutions $\{z_n\}$ satisfying

$$\sum_{i=1}^{m} A_i \partial_i z_n = \phi \tag{2.5.1}$$

and

$$z_n(x) \in K_n \tag{2.5.2}$$

for all $n$ for almost all $x$ and $K_n$ is a subset of the *state space* $\mathbb{R}^d$ containing $K$ for all $n$, when can we assert that there exists a $z$ satisfying (2.1.3) and (2.1.4). But any definitive answer to this question is so hopeless a task that the question is almost a nonsense one. We posed this question in this manner for another purpose to which we will return soon. But we can ask another question: What if instead of being given a sequence, we are given an infinite number of such sequences? Would it be possible to answer this question then?

Let us rephrase and refine the question once more. Suppose it is known beforehand that (2.5.1) has a very large set of solutions, so that given a set $\tilde{K}$, from a quite general class of sets containing $K$, we can find a solution to (2.5.1) taking values in $\tilde{K}$. Also, we know that given one such solution $z$ taking values in $\tilde{K}$, we can modify it to obtain a $\tilde{z}$, another solution of (2.5.1), which still takes values in $\tilde{K}$, but whose distribution of values is different from that of $z$. In other words, if we know that there is set $\mathcal{U}$ such that it is possible to find solutions of (2.5.1) taking values in $\mathcal{U}$ and it is possible to push the distribution of its values inside $\mathcal{U}$ without ever leaving the solution set of (2.5.1), can we solve (2.5.1) and (2.5.2) ? We will see that under appropriate assumptions, indeed we can.

Before delving into the details of how it can be done, which we will undertake in the next chapter, we shall look deeper once again to get an idea what is that we are actually asking. From the theorems in this chapter, we already know there is something regarding convexity is involved there. Recalling (1) of Theorem (2.2.3), we can already see this. Also, in the case with some information on the derivatives, $\Lambda$-convexity plays a role. In the case without any information on the derivatives we saw if we start with solutions taking values in $K$, the limit of those solutions will land up taking values in the closed convex hull of $K$. Some information on the derivatives will enable us to replace convexity by some more general and weaker notion of convexity. Now since the questions we are asking in this section is in a way, reverse in spirit to what we asked there, we now ask if we start with solutions taking values in the appropriate convex hulls of $K$, is it possible to land up, in the limit, in $K$ ? The solutions we will be starting with, i.e solutions taking values in appropriate convex hulls of $K$ is usually referred to as the solution of the '*relaxed*' problem, as this is reminiscent of '*relaxation*' in calculus of variations and optimal control.

Since $K$ is the set of convex extreme points of its convex hull ( and this fact remains true even if we replace convexity by other notions of convexity in most cases, but not all, lamination convexity being a notable exception), what we are asking is actually this:

If we know the solution set of (2.5.1) is so large that we can have rich supply of solutions taking values in a convex set $A$ ( convex hull of $K$ , or interior of convex hull, in order to get an open set ) and we can push around their value distributions in the convex set in almost arbitrary manner, can we conclude that there must also exist a solution of (2.5.1) taking values in the convex extreme points of the set $A$ ?

Clearly, such a question is related to the following question:

When can we approximate a function (in a suitable topology ) taking values in the set of extreme points of a convex set $A$ by functions taking values in the interior of the convex set $A$?

This question is indeed the key question in the method of convex integration. Gromov worked in the context of jet-spaces and his choice of topology can be described in his own terminology by the term " fine $C_0$-approximation ". But we will not delve deeper into that. Interested reader may find enriching discussions in [17]. The point we want to make is this, there is already adequate hint that the methodology of convex integration is going to be a key tool in our pursuit. In the next chapter, we will see these ideas being put to work.

# CHAPTER 3

# DIFFERENTIAL INCLUSION APPROACH

## *3.1 Basic setup*

Many nonlinear PDEs can be written as a *system of linear PDEs* and a *nonlinear pointwise constitutive relations.* The crucial observation here is that in most cases, the linear system of PDEs is *underdetermined* and in practice, it is very easy to construct enough number of solutions to the linear system of *balance laws.* If the system of balance laws admit a *potential*, then solving the nonlinear equation boils down to solving a *differential inclusion* problem. Tartar's compensated compactness framework, which roughly speaking, is plane wave analysis, localized in physical space, in combination with Gromov's *convex integration* or with *Baire category* arguments provides a well understood mechanism for generating irregular, oscillatory solutions of differential inclusion problems. We have already seen Tartar's framework in the previous chapter. In the present chapter we will be concerned with how that can help us to solve certain differential inclusion problems, which, in turn, will be used to solve certain nonlinear PDEs.

We start by briefly recalling Tartar's framework for the analysis of oscillations in systems of linear partial differential equations coupled with nonlinear pointwise constraints. As in chapter 2, we consider nonlinear PDEs that can be expressed as a system of linear PDEs (*balance Laws*)

$$\sum_{i=1}^{m} A_i \partial_i z = 0 \tag{3.1.1}$$

coupled with a pointwise nonlinear constraint (*constitutive relations*)

$$z(x) \in K \subset \mathbb{R}^d \quad a.e \tag{3.1.2}$$

where $z : \Omega \subset \mathbb{R}^m \to \mathbb{R}^d$ is the unknown state variable. The idea is then to consider *plane wave* solutions to (3.1.1), that is, solutions of the form

$$z(x) = \lambda h(x.\xi) \tag{3.1.3}$$

where $h : \mathbb{R} \to \mathbb{R}$. The *wave cone* $\Lambda$ is given by the states $\lambda \in \mathbb{R}^d$ such that for any choice of the profile $h$ the function (3.1.3) solves (3.1.1) , that is,

$$\Lambda := \left\{ \lambda \in \mathbb{R}^d : \exists \xi \in \mathbb{R}^m \backslash \{0\} \quad with \quad \sum_{i=1}^{m} \xi_i A_i \lambda = 0 \right\}. \tag{3.1.4}$$

Tartar also introduced a more general object, the *oscillation variety* $\mathcal{V}$, defined by,

$$\mathcal{V} := \left\{ (\lambda, \xi) \in \mathbb{R}^d \times \mathbb{R}^m \backslash \{0\} : \sum_{i=1}^{m} \xi_i A_i \lambda = 0 \right\}. \tag{3.1.5}$$

It is clear that the wave cone $\Lambda$ is just the projection of the oscillation variety $\mathcal{V}$ on the physical space. .The oscillatory behavior of the solutions to the nonlinear problem is then determined by the compatibility of the set $K$ with the cone $\Lambda$.

Now, if there exists a *potential* $E$ and a matrix of partial differential operators $P(D)$ such that $z = P(D)E$ identically solves (3.1.1), that is,

$$\sum_{i=1}^{m} A_i \partial_i (P(D)E) = 0$$

then (3.1.1) and (3.1.2) together reduces to

$$P(D)E \in K \subset \mathbb{R}^d \quad a.e \ , \tag{3.1.6}$$

which is a *differential inclusion* problem, where $E$ is usually a tensor and $P(D)$ is a matrix of partial differential operators, that is, each row of $P(D)$ is a partial differential operator.

Before proceeding further, in order to convince ourselves, I shall present some examples to clarify the preceding discussions.

**Example 3.1.1. Scalar conservation laws in one space dimension**

Consider

$$u_t + (f(u))_x = 0 \tag{3.1.7}$$

where $u : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ is the unknown and $f : \mathbb{R} \to \mathbb{R}$ is the flux. For the time being, we ignore the question of initial data. Now putting $v = f(u)$, $U = (v, u)$ and $y = (t, x)$, we see that the equation is equivalent to,

$$div_y U = 0 \tag{3.1.8}$$

$$U(y) \in K \tag{3.1.9}$$

where $K = Gr(f)$, the graph of $f$, is the constraint. Substituting plane waves $U(y) = (a, b)h(y \cdot \xi)$, that is,

$$U(t, x) = (a, b)h((t, x) \cdot (\xi_1, \xi_2)) = (ah(t\xi_1 + x\xi_2), bh(t\xi_1 + x\xi_2)) \ ,$$

where $\xi_1$ and $\xi_2$ are the Fourier variables corresponding to $t$ and $x$ respectively, into the balance law, we deduce, if $(a, b) \in \Lambda$, the wave cone, then $a\xi_1 + b\xi_2 = 0$. Now since for any $(a, b) \in \mathbb{R}^2$, there exists $\xi_1, \xi_2$, not both zero, such that $a\xi_1 + b\xi_2 = 0$, hence $\Lambda = \mathbb{R}^2$ in this example. This example can also be easily generalized to higher dimensions.

**Example 3.1.2. 2 × 2 elliptic system**

Let $\Omega \in \mathbb{R}^2$ be a disc and let $u : \Omega \to \mathbb{R}^2$. Consider the functional $I(u) = \int_\Omega F(\nabla u(x))\mathrm{d}x$, where $F$ is a smooth, strongly quasiconvex function [1] with uniformly bounded second derivative on $M^{2\times 2}$, the space of all real $2 \times 2$ matrices.
The Euler-Lagrange equation for $I$ is,

$$divDF(\nabla u) = 0 \tag{3.1.10}$$

[2] We denote by $J$ the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The condition that the $2 \times 2$ tensor $DF(\nabla u)$ be divergence-free is equivalent to the condition that $DF(\nabla u)J$ is the gradient of a function $\tilde{u} : \Omega \to \mathbb{R}^2$. We now

[1] A function $f : M^{m\times n} \to \mathbb{R}$ is *quasiconvex* if $\int_\Omega (f(A+\nabla\phi) - f(A)) \geq 0$ for each $A \in M^{m\times n}$ and each smooth, compactly supported $\phi : \Omega \to \mathbb{R}^m$. $f : M^{m\times n} \to \mathbb{R}$ is *strongly quasiconvex* if there exists $\gamma > 0$ such that $\int_\Omega (f(A + \nabla\phi) - f(A)) \geq \gamma \int_\Omega |\nabla\phi|^2$ for each $A \in M^{m\times n}$ and each smooth, compactly supported $\phi : \Omega \to \mathbb{R}^m$.

introduce $w : \Omega \to \mathbb{R}^4$ by $w = \begin{pmatrix} u \\ \tilde{u} \end{pmatrix}$. We also set $K$ to be the set of all $4 \times 2$ matrices of the form $\begin{pmatrix} X \\ DF(X)J \end{pmatrix}$, where $X$ runs through all $2 \times 2$ matrices.
Then (3.1.10) is equivalent to,

$$\nabla w \in K. \tag{3.1.11}$$

**Example 3.1.3. Incompressible Euler equation**

Consider the incompressible Euler equation in n-space dimensions,

$$\partial_t v + div(v \otimes v) + \nabla p = 0 \tag{3.1.12}$$

$$div\ v = 0 \tag{3.1.13}$$

We define $U = \begin{pmatrix} u + qI_n & v \\ v & 0 \end{pmatrix}$ with $q = p + \frac{1}{n}|v|^2$, $u = v \otimes v - \frac{1}{n}|v|^2 I_n$. We also set $y = (x, t)$. Then (3.1.12) is equivalent to ,

$$div_y U = 0, \tag{3.1.14}$$

[2] along with the set of constraint $K$ defined by the relations between $u$ and $v$ and $p$ and $q$.

We shall now proceed to the theory of differential inclusions along with main methods of solutions and their interesting features.

## *3.2 Differential Inclusions*

The theory of differential inclusions have a long history though their application to nonlinear PDEs are rather new. Cellina considered ordinary differential inclusions related to problems in optimal control (Cf. [3], [1]). Gromov studied extensively partial differential inclusions in his classic 'Partial differential relations' [17]. An excellent reference for both differential inclusions and their applications to nonlinear PDEs can be found in [8], [7]. We shall illustrate the basic idea of solving a differential inclusion by using the most well-studied example of partial differential inclusions, namely the *gradient inclusion.* The presentation in the following section is influenced by works of Kirchheim and Sychev (Cf. [32], [33], [20], [19]).

Consider the basic gradient inclusion problem with Dirichlet boundary data, where we will be looking for Lipschitz solutions,

$$\begin{aligned} \nabla u(x) &\in K \quad \text{for } a.e.\ x \ \text{ in } \ \Omega \\ u(x) &= f(x) \quad \text{for all } x \ \text{ in } \partial\Omega \end{aligned} \tag{3.2.1}$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz, $K \subset \mathcal{M}^{m \times n}$ is closed and bounded. In general, it is difficult to write down a solution at one strike. So we will be working in an auxiliary 'working space' and try to build the solution step by step. We first choose a set $\mathcal{U} \in \mathcal{M}^{m \times n}$ of matrices such that

(a) we can realize the boundary data $f$ with a "simple map" from $\Omega$ to $\mathbb{R}^m$ having gradients in $\mathcal{U}$ almost everywhere.

(b) "simple maps" with a gradient in $\mathcal{U}$ allow a gradual and local modification of their gradient distribution, which moves this distribution inside $\mathcal{U}$ towards $K$.

[2] the divergence of a matrix means taking row-wise divergence

The terminology “simple map” is a bit vague and it also slightly differs in (a) and (b). In (b), it usually means an affine map, whereas in (a), it usually means a (countably) piecewise affine map or almost everywhere locally affine Lipschitz map.

However, in other differential inclusion problems, of course the terms “ simple maps” can mean different types of maps. But the basic methodology is still the same. Choose a set $\mathcal{U}$, usually referred to as the *universum*, such that it is possible to construct “nice” ‘approximate’ solutions of the partial differential inclusions, that is, it is possible to construct easily a large number of ‘special’ maps satisfying the inclusion with $K$ replaced by the set $\mathcal{U}$. What do we mean by ‘special’ maps? This means maps for which we can find a method to modify the partial differential distribution, that is, the distribution of $P(D)E$, ( in case of gradient inclusion, this just means gradient distribution ) inside $\mathcal{U}$ to ‘gradually approach’ $K$. Of course, we have to specify a topology to formalize the terminology ‘gradually approach’. For gradient inclusions, mostly this will mean $dist(\nabla u_j, K) \to 0$ in $L^1(\Omega)$, where $\{u_j\}$ is the sequence of approximate solutions, that is, $\nabla u_j \in \mathcal{U}$ for all $j$.

Once we are able to find such a set $\mathcal{U}$ and the approximate solutions, we can solve the differential inclusion problem by using convex integration or Baire Category arguments. Next we discuss these methods one by one and show how they are used to produce solution of the differential inclusion problems. For simplicity of presentation, we will focus on the gradient inclusion problem (3.2.1).

## 3.3 Convex Integration

We start with the convex integration method, first introduced by Gromov in the far reaching generalization of Nash’s work on embedding problems. Müller and Sverak adapted the method to the framework of Lipschitz solutions. A typical result in the spirit of convex integration is the following:

**Theorem 3.3.1.** *Let $S$ be a bounded subset of $\mathcal{M}^{m\times n}$ and let $K$ be a compact subset of $\mathcal{M}^{m\times n}$. Assume that for every $A \in S$ and every $\epsilon > 0$ there is a piecewise affine function $\phi \in l_A + W_0^{1,\infty}(\Omega;\mathbb{R}^m)$ with the properties:*

*1) $D\phi \in (S \cup K)$ a.e. in $\Omega$;*

*2) $\|dist(D\phi, K)\|_{L^1(\Omega)} \leqslant \epsilon \; meas(\Omega)$.*

*Then for each piecewise affine function $f \in W_0^{1,\infty}(\Omega;\mathbb{R}^m)$ with $Df \in S \cup K$ a.e. in $\Omega$ the problem (3.2.1) has a solution. Moreover, each $\epsilon$-neighborhood of $f$ in $L^\infty(\Omega;\mathbb{R}^m)$ norm contains a solution of this problem.*

Here the notation $l_A + W_0^{1,\infty}(\Omega;\mathbb{R}^m)$ means the set of Lipschitz functions $u$ such that $u = l_A$ on $\partial\Omega$ where $l_A$ is a linear function with gradient equal to $A$.

**Remark 3.3.1.** The assumptions on $S$ and $K$ do not depend on $\Omega$. First we can assume that $0 \in \Omega$. Given a bounded open set $\tilde{\Omega}$ and $\epsilon > 0$ we can consider a decomposition

$$\tilde{\Omega} = \bigcup_{i=1}^{\infty} \overline{(x_i + \epsilon_i \Omega)} \cup N$$

where $0 < \epsilon_i < \epsilon$ and $meas \; N = 0$. We can define a new function $\tilde{\phi} \in l_A + W_0^{1,\infty}(\tilde{\Omega};\mathbb{R}^m)$ as follows:

$$\tilde{\phi}(x) := l_A + \epsilon_i(\phi - l_A)\left(\frac{x - x_i}{\epsilon_i}\right), \qquad x \in x_i + \epsilon_i\overline{\Omega} \tag{3.3.1}$$

$$\tilde{\phi}(x) := l_A(x) \qquad x \in N$$

Note that $D\tilde{\phi}$ has the same distribution in $\tilde{\Omega}$ as $D\phi$ in $\Omega$. In particular, 1) and 2) hold for $\tilde{\phi}$ and $\tilde{\Omega}$ if the same is true for $\phi$ and $\Omega$. Moreover,

$$\|\tilde{\phi} - l_A\|_{C(\tilde{\Omega})} \leq \sup_i \epsilon_i \|\phi - l_A\|_{C(\Omega)} \leq \epsilon \|\phi - l_A\|_{C(\Omega)}.$$

Therefore $L^\infty$ norm of the perturbation can be made as small as needed without changing the distribution of the gradient. These will be useful in the proof.

*Proof.* Let $f$ be a piecewise affine function with $Df \in (S \cup K)$ a.e. in $\Omega$. We will construct a sequence of piecewise affine functions $f_j : \Omega \to \mathbb{R}^m$ such that

$$Df_j \in (S \cup K) \text{ a.e. in } \Omega, \ \|dist(Df_j, K)\|_{L^1} \to 0, \tag{3.3.2}$$

$$f_j|_{\partial\Omega} = f|_{\partial\Omega}, \tag{3.3.3}$$

$$f_j \to f_\infty \text{ in } W^{1,1}(\Omega, \mathbb{R}^m). \tag{3.3.4}$$

We take $f_1 = f$. For a fixed $j \in \mathbb{N}$ we choose a set $\Omega_j \subset\subset \Omega$ such that

$$meas\ (\Omega \backslash \Omega_j) \leq \frac{1}{2^j}\ meas\ \Omega \tag{3.3.5}$$

and $\Omega_j = \bigcup_{k=1}^{n(j)} \Omega_j^k$, where $\Omega_j^k$ are disjoint tetrahedra such that $Df_j$ is constant in each $\Omega_j^k$ and

$$c(\text{in-radius } \Omega_j^k) \geq diam\ \Omega_j^k, \quad k \in \{1, \ldots, n(j)\} \tag{3.3.6}$$

with some $c > 0$ independent of $j$ and $k$.

We assume $d_j$ is the minimum in the set of diameters of balls inscribed in the sets $\Omega_j^k,\ k \in \{1, \ldots, n(j)\}$ and $D_j$ is the maximum in the set of diameters of the sets $\Omega_j^k,\ k \in \{1, \ldots, n(j)\}$. We can also assume $D_j \to 0$ as $j \to \infty$.

Take $s_j,\ j \in \mathbb{N}$, such that $s_j \leq \frac{s_{j-1}}{2}$ and assume also that

$$\frac{1}{2}(d_j \frac{1}{2^j}) \geq s_j. \tag{3.3.7}$$

Note that in this case $\sum_{j=i+1}^{\infty} s_j \leq s_i$, $\forall i \in \mathbb{N}$.

For each fixed $j \in \mathbb{N}$ and given $k \in \{1, \ldots, n(j)\}$ we assume

$$f_{j+1} := f_j + \phi_j^k \ \text{ in } \Omega_j^k, \tag{3.3.8}$$

where $\phi_j^k \in W_0^{1,\infty}(\Omega_j^k; \mathbb{R}^m)$ is a piecewise affine function such that

$$Df_j + D\phi_j^k \in (S \cup K) \text{ a.e. in } \Omega_j^k, \tag{3.3.9}$$

$$\|dist(Df_j + D\phi_j^k; K)\|_{L^1(\Omega_j^k)} \leq \frac{1}{2^j} meas\ \Omega_j^k, \tag{3.3.10}$$

$$\|\phi_j^k\|_{L^\infty()\Omega_j^k} \leq \frac{s_j}{2}. \tag{3.3.11}$$

In the remaining part $\Omega \backslash \bigcup_{k=1}^{n(j)} \Omega_j^k$, we assume $f_{j+1} = f_j$.

Note that in the case $\Omega_j^k = \Omega$ (in this case $f_j$ is affine in $\Omega$ ) assumptions of Theorem 3.3.1 imply that there exists a function $\phi$ which satisfies both the requirements (3.3.9) and (3.3.11). Then we can apply the arguments of Remark 3.3.1 to construct a perturbation $\phi_j^k \in W_0^{1,\infty}(\Omega_j^k; \mathbb{R}^m)$ meeting all the requirements (3.3.9)-(3.3.11). Note that (3.3.2) and (3.3.3) also hold. Hence such functions $\phi_j^k$ satisfying the requirements (3.3.9)-(3.3.11) exist.

To prove the theorem now we only need to show the convergence of $f_j$ in $W^{1,1}$-norm. Note that $f_j$ is a Cauchy sequence in $L^\infty$-norm in view of (3.3.11) and convergence of the series $\sum_{i=1}^\infty s_i$. Hence $f_j \overset{*}{\rightharpoonup} f_\infty$ in $W^{1,\infty}(\Omega;\mathbb{R}^m)$.

We have

$$\begin{aligned}\|Df_j - Df_\infty\|_{L^1(\Omega)} \leq \|Df_j\|_{L^1(\Omega\setminus\Omega_j)} + &\|Df_\infty\|_{L^1(\Omega\setminus\Omega_j)} \\ &+ \sum_{k=1}^{n(j)} \|Df_j - Df_\infty\|_{L^1(\Omega_j^k)}.\end{aligned} \tag{3.3.12}$$

The first two terms on the right hand side converge to 0 as $j \to \infty$. To establish the convergence to zero of the third one we define $f_\infty^j = f_\infty$ on the vertices of $\Omega_j^k$ and assume that $f_\infty^j$ is affine in each $\Omega_j^k,\ k \in \mathbb{N}$.

Since almost all points of $\Omega$ are both Lebesgue points of $Df_\infty$ and points of classical differentiability of $f_\infty$, given such a point $x_0$ and $\epsilon > 0$ one can find $\delta > 0$ such that if $x_0 \in \Omega_j^k$ and diam $\Omega_j^k \leq \delta$ then,

$$\begin{aligned}|Df_\infty(x_0) - Df_\infty^j(x)| \leq \epsilon,\ x \in \Omega_j^k, \\ \|Df_\infty - Df_\infty^j\|_{L^1(\Omega_j^k)} \leq \epsilon\ meas\ \Omega_j^k.\end{aligned}$$

Hence it follows that

$$\|f_\infty^k - f_\infty\|_{W^{1,1}(\cup_k \Omega_j^k;\mathbb{R}^m)} \to 0,\ j \to \infty \tag{3.3.13}$$

Since the restriction of $f_j$,$f_\infty^j$ to $\Omega_j^k$ are affine functions the maximum of the difference $|f_j - f_\infty^j|$ in $\Omega_j^k$ is attained at the vertices. Therefore

$$\|f_j - f_{infty}^j\|_{L^\infty(\Omega_j^k)} \leq \|f_j - f_\infty\|_{L^\infty(\Omega_j^k)} \leq s_j,\ k \in \{1,\ldots,n(j)\} \tag{3.3.14}$$

In each $\Omega_j^k$, $k \in \{1,\ldots,n(j)\}$, the functions $Df_j$, $Df_\infty^j$ are constant and

$$|Df_j - Df_\infty^j| \leq \frac{1}{2^j} \tag{3.3.15}$$

-otherwise (3.3.7) implies $\|f_j - f_\infty^j\|_{L^\infty(\Omega_j^k)} > s_j$ which contradicts (3.3.14).

The inequalities (3.3.12),(3.3.15) and the convergence (3.3.13) imply the convergence

$$\|f_j - f_\infty\|_{W^{1,1}(\Omega;\mathbb{R}^m)} \to 0,\ j \to \infty. \tag{3.3.16}$$

By (3.3.2) the function $f_\infty$ is a solution of the differential inclusion (3.2.1). To show that given $\epsilon > 0$ we can find $f_\infty$ satisfying both (3.2.1) and the inequality $\|f - f_\infty\|_{L^\infty} \leq \epsilon$ one should take $s_j,\ j \in \mathbb{N}$, such that $\sum_{j=1}^\infty s_j \leq \epsilon$.

This completes the proof. □

Now we shall try to give an idea of how Theorem (3.3.1) is used. The original scheme by Gromov is the following. Let $U \in \mathcal{M}^{m\times n}$ be an open set. The lamination convex hull of $U$ is,

$$\begin{aligned}U^{lc} = \cup_{k=0}^\infty U^{(k)}, where\ U^{(0)} = U, \\ U^{(k+1)} = U^{(k)} \cup \{[A,B] : A, B \in U^{(k)},\ rank(B-A) = 1\}.\end{aligned}$$

**Definition 3.3.2** (Gromov)**.** A sequence of open sets $U_i \in \mathcal{M}^{m\times n}$ is an in-approximation of a set $K \in \mathcal{M}^{m\times n}$ if

(1) the $U_i$ are uniformly bounded;

(2) $U_i \subset U_{i+1}^{lc}$;

(3) $U_i \to K$ in the following sense: if $F_i \in U_i$ and $F_i \to F$ then $F \in K$.

**Theorem 3.3.3** (Gromov)**.** *Suppose that $\{U_i\}$ is an in-approximation of a compact set $K \in \mathcal{M}^{m\times n}$ and that $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is $C^1$(or piecewise $C^1$) and satisfies $f \in U_1$ a.e. Then the problem (3.2.1) has a solution.*

Also Müller-Šverák also used another scheme of approximating the set K, where the in-approximations are replaced by more general rank-1-in-approximations, called rc-in-approximations, which is defined analogously to in-approximation but using rank-1 convex hulls rather than lamination convex hulls.

Theorem (3.3.3) and its rc-in-approximation version are actually both particular cases of Theorem (3.3.1) where the set $S$ is taken as the union of the elements of the sets $U_i$s. The proof of this fact relies on the following lemma:

**Lemma 3.3.4.** *Assume that $c \in \mathbb{R}^m$ and assume that $b \in \mathbb{R}^n$. Let $b_1 = t_1 b,\ b_2 = t_2 b$, where $t_2 < 0 < t_1$, and let $b_1, \ldots, b_q$ be extreme points of a compact convex set with $0 \in int\ co\{b_1, \ldots, b_q\}$. Define $B_i := c \otimes b_i,\ i \in \{1, \ldots, q\}$. Then for each $\epsilon > 0$ there exists a piecewise affine function $\phi \in W_0^{1,\infty}(\Omega; \mathbb{R}^m)$ such that,*

$$meas\ \{x \in \Omega : D\phi(x) = B_1\ or\ D\phi(x) = B_2\} \geq meas(\Omega) - \epsilon, \tag{3.3.17}$$

$$D\phi \in \{B_1, \ldots, B_q\}\ a.e.\ in\ \Omega,\ \|\phi\|_{C(\Omega)} \leq \epsilon. \tag{3.3.18}$$

*Proof.* It is enough to prove the lemma for the scalar valued case $m = 1$(with $c = 1$), since if (3.3.17) and (3.3.18) hold for a function $\psi \in W_0^{1,\infty}(\Omega)$ then we can define the function $\phi : \Omega \to \mathbb{R}^m$ as $\phi_i = c_i\psi,\ i \in \{1, \ldots, m\}$. Then $D\phi = c \otimes D\psi$ and the result holds for general case.

To prove the lemma in the scalar valued case consider first the extreme points $v_1, \ldots, v_q$ of a compact subset of $\mathbb{R}^n$ with $0 \in int\ co\{v_1, \ldots, v_q\}$. Consider the function

$$w_s(.) = max_{v\in\{v_1,\ldots,v_q\}}\langle v, .\rangle - s,\ s > 0. \tag{3.3.19}$$

It is clear that $w_s(.)$ is a Lipschitz function such that $Dw_s \in \{v_1, \ldots, v_q\}$ a.e and $w_s(.) = 0$ in $\partial P_s$, where $P_s$ are polyhedrons such that $P_s = sP_1$. We can decompose $\Omega$ into domains $\Omega_i := x_i + sP_1$ with $s_i \leq \epsilon$, $i \in \mathbb{N}$, and a set $N$ of measure zero, i.e. $\Omega = \cup_{i\in\mathbb{N}}(x_i + s_i\overline{P_1}) \cup N$. Define $u(x) = w_{s_i}(x - x_i)$ for $x \in x_i + siP_1,\ i \in \mathbb{N},\ u = 0$ otherwise. Then $u \in W_0^{1,\infty}(\Omega),\ Du \in \{v_1, \ldots, v_q\}$ a.e. in $\Omega$.

We can take $v_1 = b_1,\ v_2 = b_2$ and $v_i \in B(b_1, \epsilon) \cap int\ co\{b_1, \ldots, b_q\},\ i \in \{3, \ldots, q\}$. Then we can perturb the function $u$ in each set $\Omega_i := \{x \in \Omega : Du(x) = v_i\},\ i \in \{3, \ldots, q\}$ in such a way that the perturbation $\phi_\epsilon$ has the property $D\phi_\epsilon \in \{b_1, \ldots, b_q\}$. We can do this since $v_i \in int\ co\{b_1, \ldots, b_q\}$ and the construction in (3.3.19) can be applied to find a piecewise affine function $f_i \in W_0^{1,\infty}(\Omega_i)$ such that $Df_i \in \{b_1 - v_i, \ldots, b_q - v_i\}$. Then, the function $l_{v_i} + f_i$ is the perturbation in question.

Note that,

$$meas\ \{x \in \Omega_i : Df_i(x) \neq b_1 - v_i\} \to 0,\ \text{as } \epsilon \to 0,\ \forall i \in \{3, \ldots, q\}.$$

Then

$$meas\ \{x \in \Omega : D\phi_\epsilon \notin \{b_1, b_2\}\} \to 0,\ \text{as } \epsilon \to 0.$$

Note also that

$$\|\phi_\epsilon\|_{C(\Omega)} \leq \sup_{i\in\mathbb{N}} s_i \leq \epsilon.$$

Therefore $\|\phi\|_{C(\Omega)}$ can be made arbitrarily small.

This proves the lemma. □

Now we proceed to show that Theorem (3.3.3) is a special case of Theorem (3.3.1). We will use Lemma (3.3.4) to show that in this case $S = \cup_i U_i$. In fact, given $\epsilon > 0$ we can find $i \in \mathbb{N}$ such that for all $A \in U_i$ we have $dist_{A \in U_i}(A, K) \leq \epsilon$. Then we can apply Lemma (3.3.4) iteratively to perturb the original linear boundary data $l_F$ with $F \in U_1$ in such a way that the gradient of the piecewise affine perturbation $\phi \in l_F + W_0^{1,\infty}(\Omega; \mathbb{R}^m)$ stays in $S$ and it stays in $U_i$ in the set with measure exceeding $meas(\Omega) - \epsilon$. Then

$$\int_\Omega dist(D\phi(x), K) \mathrm{d}x \leq \epsilon \; meas(\Omega) + \epsilon \; diam(S) \to 0 \text{ as } \epsilon \to 0.$$

This proves that $S$ and $K$ satisfy the condition of Theorem (3.3.1). Since $U_1$ is open we can also approximate each piecewise $C^1$ function $f$ with $Df \in U_1$ a.e. by piecewise affine functions such that they assume the same values at $\partial\Omega$ and their gradients stay in $U_1$ a.e.

Theorem (3.3.1) reduces the problem of solving gradient inclusion problems, and hence, any nonlinear partial differential equation or system that can be reduced to a gradient inclusion problem, to finding appropriate approximations of the set $K$. It should be mentioned that which type of approximation will suit the problem at hand depends mainly on the geometry of $K$, in more precise terms, the largeness of different convex hulls of $K$. For example, where the lamination convex hull of $K$ is large enough, we usually look for in-approximations, whereas when lamination convex hull is not large enough, but the rank-1-convex hull is, we look for rc-in-approximations. Of course, finding such approximations is far from trivial and examples of implementation of these ideas in particular cases and a somewhat detailed discussion about the choice of appropriate approximation and its relation with different convex hulls can be found in [25]. Further interesting material in these directions can be found in [26], [24], [21].

The primary reason for treating only the case of gradient inclusion is ease of presentation of ideas. Firstly, gradient inclusion is the most studied case among partial differential inclusions and hence satisfactory theorems like Theorem (3.3.1) are available. However, the basic setup that is described in Section (3.2) is same in the case of other types of differential inclusions too. However, necessary modifications are neither always easy nor trivial. We shall present an example of a different differential inclusion in the context of application to Euler equations for incompressible fluids in Section (3.6)

We shall now proceed to outline the basic prototype of Baire Category arguments in the following section.

## 3.4 Baire Category method

Once again we will limit our attention to gradient inclusions, i.e we are considering the problem (3.2.1). The core of the Baire Category method lies in the fundamental observation, which although simple, has been completely overlooked in this context for a long time, that the gradient map $u \in (W^{1,\infty}, \|.\|_{L^\infty}) \to \nabla u \in L^p,\ p < \infty$ has a rather surprising continuity property in the framework of Baire Category. The continuity looks quite counterintuitive in view of the linear theory and reflects the underlying nonlinearity of th situation, which is manifested by the boundedness of the universum $\mathcal{U}$. The abovementioned continuity property is summarized in the following theorem.

**Theorem 3.4.1.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set and let $\mathcal{F} \subset Lip(\Omega; \mathbb{R}^m)$ equipped with the supremum norm $\|.\|_\infty$ be complete. Then for any $p < \infty$ the map $\nabla : f \to \nabla f$ is Baire class one maps from $(\mathcal{F}, \|.\|_\infty)$ into $L^p(\Omega; \mathcal{M}^{m\times n})$, i.e. is the pointwise limit of a sequence $\Delta_k : (\mathcal{F}, \|.\|_\infty) \to L^p(\Omega; \mathcal{M}^{m\times n})$ of continuous maps.*

*In particular, the typical $f$ (i.e. all $f \in \mathcal{F}$ except those from a set of first Baire category in $\mathcal{F}$) is a point of continuity of the map $\nabla$.*

*Proof.* For $k \geq 1$ and $f \in \mathcal{F}$ we define, denoting by $e_j$ the $j$-th standard unit vector,

$$(\Delta_k(f))(x)_{i,j} = \begin{cases} k(f_i(x + \frac{1}{k}e_j) - f_i(x)) & \text{if } dist(x, \mathbb{R}^n \backslash \Omega) > \frac{1}{k} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\|\Delta_k(f)\|_\infty \leq Lip(f)$ for all $f$ and $k$, moreover, if $f$ is differentiable at $x$ then $\Delta_k(f)(x) \to \nabla f(x)$ as $k \to \infty$. By Rademacher's theorem and Lebesgue's dominated convergence theorem, we have $\lim_k \Delta_k(f) = \nabla f$ in $L^p(\Omega; \mathcal{M}^{m\times n})$ as $k \to \infty$. On the other hand, each $\Delta_k$ is a linear operator which has nor at most $2k|\Omega|^{\frac{1}{p}}$ when acting from $(\mathcal{F}, \|.\|_\infty)$ into $L^p(\Omega; \mathcal{M}^{m\times n})$. Thus $\nabla$ is a Baire-one map. This, together with the well-known fact that any Baire-one map from a complete metric space to any metric space has a dense $G_\delta$ set of points of continuity, the theorem follows. □

To show how this fact can be used to solve gradient inclusions, first we need a few definitions.

**Definition 3.4.2.** Given $\Omega \subset \mathbb{R}^n$ bounded and open, $\mathcal{U} \subset \mathcal{M}^{m\times n}$ bounded but not necessarily open, we put

$$\mathcal{P}(\Omega, \mathcal{U}) = \{u \in Lip(\Omega, \mathbb{R}^m) : u \text{ is piecewise affine and } \nabla u \in \mathcal{U} \text{ a.e. in } \Omega\}.$$

If there is in addition a map $f$ from a superset of $\partial\Omega$ into $\mathbb{R}^m$ given, then we define the space

$$\mathcal{P}(\Omega, \mathcal{U}, f) = \{u \in \mathcal{P}(\Omega, \mathcal{U}) : u(x) = f(x) \text{ for all } x \in \partial\Omega\}.$$

**Definition 3.4.3.** Let $\mathcal{U}$, $K \subset \mathcal{M}^{m\times n}$ be given. We say that gradients in $\mathcal{U}$ are stable only near $K$ if $\mathcal{U}$ is bounded, $K$ is closed and for each $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that for all $A \in \mathcal{U}$ with $dist(A, K) > \epsilon$ there exists a piecewise affine $\phi \in Lip(\mathbb{R}^n, \mathbb{R}^m)$ with bounded support which satisfies

- $A + \nabla\phi(x) \in K_A$ for a.e $x \in \Omega$, where $K_A \subset \mathcal{U}$
- $\int |\nabla\phi(x)| > \delta \; meas \; (supp \; \phi)$

Now we can state a result which is typical of the Baire category method.

**Theorem 3.4.4.** *Suppose that gradients in $\mathcal{U}$ are stable only near $K$. Given any $A \in \mathcal{U}$ and $\Omega \subset \mathbb{R}^n$ bounded open set, let $\mathcal{P}$ be the space of all piecewise affine Lipschitz $u : \Omega \to \mathbb{R}^m$ with $u|_{\partial\Omega} = A$ and $\nabla u(x) \in \mathcal{U}$ a.e.*

*Then the typical $u \in \overline{\mathcal{P}(\Omega, \mathcal{U}, f)}^\infty$ ( in the sense of Baire category ) is a solution of (3.2.1).*

Here $\overline{\mathcal{P}(\Omega, \mathcal{U}, f)}^\infty$ means the closure of the space $\mathcal{P}(\Omega, \mathcal{U}, f)$ in $L^\infty$-norm topology.

*Proof.* Using Theorem (3.4.1), it is enough to show that if $u$ is a point of continuity of the gradient map restricted to $\overline{\mathcal{P}(\Omega, \mathcal{U}, f)}^\infty$, then $\nabla u(x) \in K$ a.e $x$ in $\Omega$. Suppose for contradiction that $u$ is point of continuity of the gradient map but $\nabla u \notin K$ on a set of positive measure. Using Lusin's theorem we can find a compact set $\tilde{C} \subset \Omega$ and an $\epsilon > 0$ such that

- $meas \; \tilde{C} > \epsilon$, $\nabla u|_{\tilde{C}}$ is continuous, and
- dist $(\nabla u(x), \; K) > \epsilon$ for all $x \in \tilde{C}$.

We pick an $\eta > 0$ such that $\|\nabla u - \nabla v\|_{L^1} < \epsilon\frac{\delta(\epsilon)}{4}$ whenever $v$ belongs to the set $B_\infty(u, \eta) \cap \overline{\mathcal{P}(\Omega, \mathcal{U}, f)}^\infty$. By definition of $\overline{\mathcal{P}(\Omega, \mathcal{U}, f)}^\infty$ there is a sequence $u_k \in \mathcal{P}(\Omega, \mathcal{U}, f)$ such that $u_k \to u$ in $L^\infty$ norm. Since $u$ is a point of continuity of the gradient map, by passing to a subsequence if necessary we have $\nabla u_k(x) \to u(x)$ a.e. Hence, we find $k_0 \in \mathbb{N}$ and another compact set $\hat{C} \subset \Omega$ with *meas* $\hat{C} > \epsilon$ and $\mathrm{dist}(\nabla u_{k_0},\ K) > \epsilon$ for all $x \in \hat{C}$. We can also assume that $\|u - u_{k_0}\|_\infty < \frac{\eta}{2}$. As $u_{k_0}$ is piecewise affine, there are disjoint open subsets $\{G_i\}_1^\infty$ of $\Omega$ such that *meas* $(\Omega\backslash \cup G_i) = 0$ and $u_{k_0}|_{G_i}$ is affine with gradient $A_i \in \mathcal{U}$. We choose $I \subset \mathbb{N}$ finite such that $\hat{C} \cap G_i \neq \emptyset$ for each $i \in I$ and that $\sum_{i\in I}$ *meas* $G_i > \epsilon$. Due o the definition of $\delta(\epsilon)$, we can find for each $i \in I$ a piecewise affine Lipschitz map $\phi_i : G_i \to B(0, \frac{\eta}{2}; \mathbb{R}^m)$ with $\int_{G_i} |\nabla\phi_i(x)|\mathrm{d}x > \delta(\epsilon)$ *meas* $G_i$, $\phi_i|_{G_i} = 0$ and $A_i + \nabla\phi_i(x) \in K_i$ a.e in $G_i$, where $K_i \subset \mathcal{U}$. Note that to construct such $\phi_i$s we have used the arguments in Remark (3.3.1). Now it is clear that $g = u_{k_0} + \sum_{i\in I} \phi_i \in B_\infty(u,\eta) \cap \mathcal{P}(\Omega,\mathcal{U}, f)$ and the same is true for $u_{k_0}$ itself. We have, however, $\|\nabla u_{k_0} - \nabla g\|_{L^1} \geq \epsilon\delta(\epsilon)$, and this contradicts the choice of $\eta$ and establishes the theorem. □

So we have solved the gradient inclusion problem. But what exactly has been done? Apart from the continuity property, Definition (3.4.3) contains an interesting point. We shall explore these issues in the next section.

## *3.5 Brief overview of the two methods*

To be extremely brief, the basic point of the convex integration method can be given in the following:

- Controlled $L^\infty$ convergence gives strong $W^{1,1}$ convergence

So what convex integration method does is to :

- Start with a Lipschitz function having gradients in $\mathcal{U}$ almost everywhere, but having the prescribed boundary values.
- Find an explicit scheme of iterative approximations, in each step perturbing the gradient of the function, without leaving $\mathcal{U}$ and without changing the boundary values such that
  - The successive approximations gets closer and closer to the target set $K$ and,
  - $L^\infty$ convergence of the successive approximation can be controlled.

This method of solving the inclusion, as can be readily seen is much more geometric in nature. To successfully carry out the steps described above, the knowledge of different hulls of the set $K$ plays a crucially important role.

The Baire category method is more functional analytic in nature. Instead of working very hard to find an explicit scheme of successive approximations, this method relies on a quite different idea. To understand this, we return to Definition (3.4.3).

What Definition (3.4.3) says is roughly the following:
If a matrix $A \in \mathcal{U}$ is away from $K$, then we can find special piecewise affine Lipschitz perturbation to modify the gradient of a function having gradients equal to $A$ almost everywhere. This can be done without changing the boundary values of the original function and *without leaving* the universum $\mathcal{U}$ . The idea is to perturb a function (say $f$) having gradients almost everywhere in $\mathcal{U}$ without leaving $\mathcal{U}$, i.e. the gradient of the perturbed function is again in $\mathcal{U}$. We then wish to iterate the process and hope that in the limit of these successive iterations will present us a function having gradients in $K$ almost everywhere. Now while we iterate our perturbation process, how does the gradient change in successive iterations? We don't know much about it, except that it never leaves the universum $\mathcal{U}$ and as far as we are concerned, the gradients can move about in $\mathcal{U}$ in quite arbitrary manner. The interesting point is that we need not care to know. Instead we imagine a kind of random walk of the gradients inside $\mathcal{U}$ as we iterate our perturbation process. Now where will the gradient land in the limit? Exactly at

those points where it got 'stuck', i.e on a set of points which has a stability with respect to the type of perturbations we are applying. Now if we know beforehand that if the gradient is away from $K$, it can easily be perturbed, i.e the gradients are not stable away from $K$ (this is in a way, what Definition (3.4.3) actually means), then in the limit, the gradient must lie in $K$ in the 'typical' case (Of course we need to make sense what 'typical' really means here, and in a sense this is the whole point, that 'typical' here means in a Baire generic sense ), which is the content of Theorem (3.4.4).

Of course, the above discussion is by no means intended to suggest that any one method is superior to the other in some respect. The scope of these two methods is not exactly the same, neither does the scope of one encompasses the other. But for most common applications, both methods work. Also both the method can be unified to a large extent, as can be readily seen by our use of Remark (3.3.1 ) in proving Theorem (3.4.4). Also the continuity property, i.e Theorem (3.4.1) is crucial and is the underlying reason for both the methods to succeed. Again, though the discussion in the previous paragraph might give the impression that using Baire category methods are easier than using convex integration, since we do not need to find explicit approximation procedure, that is only an illusion, since proving the stability property (Definition (3.4.3)) used in the hypothesis of Theorem (3.4.4) is neither an easy matter nor a trivial one.

## *3.6 Application to Euler equations*

Now we proceed to describe a very interesting application of these methods to Euler equations for incompressible fluid flow due to De Lellis and Székelyhidi Jr.. This application is interesting in its own right and also this will give us a flavor of a partial differential inclusion other than a gradient inclusion and can offer a glimpse at the modifications necessary in the basic strategy outlined in the last three sections. This section draws heavily from the work of De Lellis and Székelyhidi Jr. For the relation of this work with the Onsagar's conjecture and $h$-principle of Gromov and other related works, see [13].

Consider the incompressible Euler equation in n-space dimensions,

$$\left.\begin{aligned} \partial_t v + div(v \otimes v) + \nabla p = 0 \\ div\ v = 0 \end{aligned}\right\} \tag{3.6.1}$$

We first define the appropriate notion of weak solutions and subsolutions for (3.6.1).

**Definition 3.6.1** (Weak solutions)**.** A vectorfield $v \in L^2_{loc}(\mathbb{R}^n \times (0,T))$ is a weak solution of the incompressible Euler equations if

$$\int_0^T \int_{\mathbb{R}^n} (\partial_t \phi \cdot v + \nabla \phi : v \otimes v) \ \mathrm{d}x\mathrm{d}t = 0 \tag{3.6.2}$$

for all $\phi \in C_c^\infty(\mathbb{R}^n \times (0,T); \mathbb{R}^n)$ with $div\ \phi = 0$ and

$$\int_0^T \int_{\mathbb{R}^n} v \cdot \nabla \psi \ \mathrm{d}x\mathrm{d}t = 0 \tag{3.6.3}$$

for all $\psi \in C_c^\infty(\mathbb{R}^n \times (0,T))$.

**Definition 3.6.2** (Subsolutions)**.** Let $\overline{e} \in L^1_{loc}(\mathbb{R}^n \times (0,T))$ with $\overline{e} \geq 0$. A subsolution to the incompressible Euler equation with given kinetic energy density $\overline{e}$ is a triple

$$(v,\ u,\ q) : \mathbb{R}^n \times (0,T) \to \mathbb{R}^n \times \mathcal{S}_0^{n\times n} \times \mathbb{R}$$

with the following properties:

- $v \in L^2_{loc}$, $u \in L^1_{loc}$, $q$ is a distribution;

- $$\begin{cases} \partial_t v + div\ u + \nabla q = 0 \\ div\ v = 0, \end{cases} \quad \text{in the sense of distributions;} \tag{3.6.4}$$

- $$v \otimes v - u \leq \frac{2}{n}\bar{e}I \quad \text{a.e..} \tag{3.6.5}$$

Recalling from Example (3.1.3) , we define $U = \begin{pmatrix} u + qI_n & v \\ v & 0 \end{pmatrix}$ with $q = p + \frac{1}{n}|v|^2$, $u = v \otimes v - \frac{1}{n}|v|^2 I_n$. We also set $y = (x,t)$. Then (3.6.1) is equivalent to ,

$$div_y U = 0 \ , \tag{3.6.6}$$

along with the set of constraint $K$ defined by the relations between $u$ and $v$ and $p$ and $q$.

In Example (3.1.3) we did not show how to reduce it to a differential inclusion problem. We will do so shortly. But before that we want to point out another very interesting reason why this particular example has been chosen. This example illustrates an interesting connection between the framework of Tartar and the differential inclusion technique. The knowledge of the wave cone will be crucially used to build the approximate solutions. Also, the appropriate convex hull to use in the differential inclusion scheme will be the $\Lambda$-convex hull of the target set $K$, denoted by $K^\Lambda$, where $\Lambda$ is the wave cone corresponding to the linear differential system of balance laws.

To present the general framework, we recall first Tartar's framework again.

Consider nonlinear PDEs that can be expressed as a system of linear PDEs (*balance Laws*)

$$\sum_{i=1}^{m} A_i \partial_i z = 0 \tag{3.6.7}$$

coupled with a pointwise nonlinear constraint (*constitutive relations*)

$$z(x) \in K \subset \mathbb{R}^d \quad a.e \tag{3.6.8}$$

where $z : \mathcal{D} \subset \mathbb{R}^m \to \mathbb{R}^d$ is the unknown state variable. The idea is then to consider *plane wave* solutions to (3.6.7), that is, solutions of the form

$$z(x) = \lambda h(x.\xi) \tag{3.6.9}$$

where $h : \mathbb{R} \to \mathbb{R}$. The *wave cone* $\Lambda$ is given by the states $\lambda \in \mathbb{R}^d$ such that for any choice of the profile $h$ the function (3.6.9) solves (3.6.7) , that is,

$$\Lambda := \left\{ \lambda \in \mathbb{R}^d : \exists \xi \in \mathbb{R}^m \backslash \{0\} \quad with \quad \sum_{i=1}^{m} \xi_i A_i \lambda = 0 \right\}. \tag{3.6.10}$$

.The oscillatory behavior of the solutions to the nonlinear problem is then determined by the compatibility of the set $K$ with the cone $\Lambda$. It is this compatibility which is actually expressed in terms of the $\Lambda$-convex hull $K^\Lambda$. Modulo technical details, the subsolutions from Definition (3.6.2) are solutions $z$ of the linear relations (3.6.7) which satisfy the 'relaxed' condition $z \in K^\Lambda$ a.e.

The idea of convex integration is to reintroduce oscillations by adding suitable localized versions of (3.6.9) to the subsolutions and to obtain a solution of (3.6.7)-(3.6.8) iterating the process. The conclusion is that in a Baire generic sense, most solutions of the "relaxed system" are actually solutions of the original system.

In order to implement convex integration in this general framework, we will work in a space of subsolutions in which highly oscillatory perturbations are possible. This space of subsolutions arises from a nontrivial open set $\mathcal{U} \subset \mathbb{R}^d$ satisfying the following perturbation property:

**Lemma 3.6.3.** *There is a continuous function $\epsilon : \mathbb{R}^+ \to \mathbb{R}^+$ with $\epsilon(0) = 0$ and $\epsilon(x) > 0$ for $x \neq 0$, satisfying the following property:*
*For every $z \in \mathcal{U}$ there is a sequence of solutions $z_j \in C_c^\infty(B_1)$ of (3.6.7) such that*

- $z_j \overset{*}{\rightharpoonup} 0$ *in* $L^\infty(\mathbb{R}^m)$*;*
- $z + z_j(y) \in \mathcal{U} \quad \forall y \in \mathbb{R}^m$*;*
- $\int |z_j(y)|^2 \mathrm{d}y \geq \epsilon(dist(z, K))$*.*

Note that the properties mentioned in the lemma is reminiscent of the conditions in Definition (3.4.3).
Next, let

$$X_0 = \{z \in C_c^\infty(\mathcal{D}) : \text{(3.6.7) holds and } z(y) \in \mathcal{U} \text{ for all } y \in \mathcal{D}\}$$

and let $X$ be the closure of $X_0$ in $L^\infty(\mathcal{D})$ with respect to the weak-$*$ topology. Assuming that $K$ is bounded, the set $X$ is bounded in $L^\infty$ and hence the weak-$*$ topology is metrizable on $X$.

Now a standard covering argument together with Lemma (3.6.3) gives the following:

**Lemma 3.6.4.** *There is a continuous function $\tilde{\epsilon} : \mathbb{R}^+ \to \mathbb{R}^+$ with $\tilde{\epsilon}(0) = 0$ and $\tilde{\epsilon}(x) > 0$ for $x \neq 0$ such that, for every $z \in X_0$ there is a sequence $z_j \in X_0$ with*

$$\int_{\mathcal{D}} |z_j - z|^2 \mathrm{d}y \geq \tilde{\epsilon}\left(\int_{\mathcal{D}} dist(z(y), K)\mathrm{d}y\right) \tag{3.6.11}$$

Now it can be easily proved that the identity map from $X$ to $L^2(\mathcal{D})$ is a Baire-1 function. This fact together with Baire category arguments and Lemma (3.6.4) gives that the subset of $z \in X$ satisfying (3.6.8) is Baire-generic in $X$.

Now if the perturbation property holds for some $\mathcal{U}$, then the above argument yields weak solutions to (3.6.7)-(3.6.8) which are zero on the boundary $\partial\mathcal{D}$ in the trace sense with respect to the operator in (3.6.7). In case of evolution equations ( as in the case of Euler equations ), $\mathcal{D}$ is a space-time domain, say $\mathcal{D} = \mathbb{R}^n \times (0, T)$, and thus this argument yields weak solutions with compact time support.

We will now proceed to show how the abovementioned scheme is implemented in the case of Euler equation for incompressible fluid flow. First, we calculate the wave cone. In Example (3.1.3), we showed that the linear differential system of balance laws is equivalent to (3.1.14). We denote by $\mathcal{M}$ the set of symmetric $(n+1) \times (n+1)$ matrices $A$ such that $A_{(n+1)\times(n+1)} = 0$. Now, in view of the linear isomorphism

$$\mathbb{R}^n \times \mathcal{S}_0^n \times \mathbb{R} \ni (v, u, q) \mapsto U = \begin{pmatrix} u + qI_n & v \\ v & 0 \end{pmatrix} \in \mathcal{M}, \tag{3.6.12}$$

the wave cone corresponding to (3.1.14) can be written as

$$\Lambda = \left\{(v, u, q) \in \mathbb{R}^n \times \mathcal{S}_0^n \times \mathbb{R} : det \begin{pmatrix} u + qI_n & v \\ v & 0 \end{pmatrix} = 0\right\}. \tag{3.6.13}$$

Now we will present two lemmas which are crucial for the construction of the subsolutions. The first one is about a symmetry in the equation and the second one, in effect, reduces the problem to a differential inclusion problem.

**Lemma 3.6.5** (Galilean group symmetry). *Let $\mathcal{G}$ be the subgroup of $GL_{n+1}(\mathbb{R})$ defined by,*

$$\{A \in \mathbb{R}^{(n+1)\times(n+1)} : detA \neq 0,\ Ae_{n+1} = e_{n+1}\}. \tag{3.6.14}$$

*For every divergence free map $U : \mathbb{R}^{n+1} \to \mathcal{M}$ and every $A \in \mathcal{G}$ the map*

$$V(y) = A^t \cdot U(A^{-t}y) \cdot A \tag{3.6.15}$$

*is also a divergence free map $V : \mathbb{R}^{n+1} \to \mathcal{M}$*

*Proof.* It is easy to check that whenever $B \in \mathcal{M}$, then $A^tBA \in \mathcal{M}$ for all $A \in \mathcal{G}$. Now let $\phi \in C_c^\infty(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$ be a compactly supported test function and consider $\tilde{\phi}$ defined by

$$\tilde{\phi}(x) = A\phi(A^tx)$$

Then $\nabla\tilde{\phi}(x) = A\nabla\phi(A^tx)A^t$, and by a change of variables we obtain

$$\begin{aligned}
\int tr(V(y)\nabla\phi(y))\mathrm{d}y &= \int tr(A^tU(A^{-t}y)A\nabla\phi(y))\mathrm{d}y \\
&= \int tr(U(A^{-t}y)A\nabla\phi(y)A^t)\mathrm{d}y \\
&= \int tr(U(x)A\nabla\phi(A^tx)A^t)(detA)^{-1}\mathrm{d}x \\
&= (detA)^{-1}\int tr(U(x)\nabla\tilde{\phi}(x))\mathrm{d}x = 0,
\end{aligned}$$

since $U$ is divergence free. But this implies that $V$ is also divergence free. □

**Lemma 3.6.6** (Potential). *Let $E_{ij}^{kl} \in C^\infty(\mathbb{R}^{n+1})$ be functions for $i, j, k, l = 1, \ldots, n+1$ so that the tensor $E$ is skew-symmetric in $ij$ and $kl$, that is*

$$E_{ij}^{kl} = -E_{ij}^{lk} = -E_{ji}^{kl} = E_{ji}^{lk}\ . \tag{3.6.16}$$

*Then*

$$U_{ij} = \mathcal{L}(E) = \frac{1}{2}\sum_{k,l}\partial_{kl}^2(E_{il}^{kj} + E_{jl}^{ki}) \tag{3.6.17}$$

*is symmetric and divergence free. If in addition*

$$E_{(n+1)j}^{(n+1)i} = 0 \quad \textit{for every } i \textit{ and } j. \tag{3.6.18}$$

*then $U$ takes values in $\mathcal{M}$.*

*Proof.* Easy and direct calculation. □

Using these two lemmas, we can prove the following, which is essentially the core of the construction of De Lellis and Székelyhidi Jr. in [11]:

**Proposition 3.6.1** (Localized plane waves). *Let $a = (v_0, u_0, q_0) \in \Lambda$ with $v_0 \neq 0$ and denote by $\sigma$ the line segment joining the points $a$ and $-a$. For every $\epsilon > 0$ there exists a smooth solution $(v, u, q)$ of (3.1.14) (in view of the linear isomorphism (3.6.12)) with the properties:*

- *the support of $(v, u, q)$ is contained in $B_1(0) \subset \mathbb{R}_x^n \times \mathbb{R}_t$,*
- *the image of $(v, u, q)$ is contained in the $\epsilon$-neighborhood of $\sigma$,*

- $\int |v(x,t)| \mathrm{d}x\mathrm{d}t \geq \alpha |v_0|$,

*where $\alpha > 0$ is a dimensional constant.*

This proposition follows from the next one, again in view of the linear isomorphism (3.6.12).

**Proposition 3.6.2.** *Let $\overline{U} \in \mathcal{M}$ be such that det $\overline{U} = 0$ and $\overline{U} e_{n+1} \neq 0$, and consider the line segment $\sigma$ with endpoints $\overline{U}$ and $-\overline{U}$. Then there exists a constant $\alpha > 0$ such that For any $\epsilon > 0$ there exists a smooth divergence-free matrix field $U : \mathbb{R}^{n+1} \to \mathcal{M}$ with the properties:*

- *supp $U \subset B_1(0) \subset \mathbb{R}^n_x \times \mathbb{R}_t$,*
- *dist$(U(y), \sigma) < \epsilon$ for all $y \in B_1(0)$,*
- $\int |U(y) e_{n+1}| \mathrm{d}y \geq \alpha |\overline{U} e_{n+1}|$,

*where $\alpha > 0$ is a dimensional constant.*

*Proof.* Detailed proof can be found in [11]. We just sketch the argument emphasizing the crucial points. First note that using Lemma (3.6.5) and a standard covering/rescaling argument, it is enough to prove the proposition for the case where $\overline{U} \in \mathcal{M}$ is such that

$$\overline{U} e_1 = 0, \ \overline{U} e_{n+1} \neq 0. \tag{3.6.19}$$

Define

$$E^{i1}_{j1} = -E^{1i}_{j1} = -E^{i1}_{1j} = E^{1i}_{1j} = \overline{U}_{ij} \frac{\sin(N y_1)}{N^2}. \tag{3.6.20}$$

and all the other entries equal to 0. Now choose a smooth cut-off function $\phi$ such that

- $|\phi| \leq 1$,
- $\phi = 1$ on $B_{\frac{1}{2}}(0)$,
- *supp*$(\phi) \subset B_1(0)$,

and consider the map

$$U = \mathcal{L}(\phi E). \tag{3.6.21}$$

Clearly, $U$ is smooth and supported in $B_1(0)$. Using Lemma (3.6.6), $U$ is $\mathcal{M}$-valued and divergence free. Also,

$$U(y) = \overline{U} \sin(N y_1) \text{ for } y \in B_{\frac{1}{2}}(0),$$

and hence

$$\int |U(y) e_{n+1}| \mathrm{d}y \geq |\overline{U} e_{n+1}| \int_{B_{\frac{1}{2}}(0)} |\sin(N y_1)| \mathrm{d}y \geq 2\alpha |\overline{U} e_{n+1}|, \tag{3.6.22}$$

for large enough $N$. Also note that

$$\|U - \phi \tilde{U}\|_\infty \leq \frac{C}{N} \|\phi\|_{C^2}, \tag{3.6.23}$$

where $\tilde{U} = \mathcal{L}(E)$. Hence by choosing $N$ large enough, we can easily obtain $\|U - \phi \tilde{U}\|_\infty \leq \epsilon$. Since $|\phi| \leq 1$ and consequently the image of $\tilde{U}$ is contained in $\sigma$, this proves the proposition. □

The construction of the localized plane waves is the crucial issue in the whole matter. Rest is just a matter of carrying out the general strategy we already outlined. However, before proceeding further, we formally state our desired theorem.

**Theorem 3.6.7.** *Let $\Omega \subset \mathbb{R}^n_x \times \mathbb{R}_t$ be a bounded open domain. There exist $(v,p) \in L^\infty(\mathbb{R}^n_x \times \mathbb{R}_t)$ solving the Euler equations*

$$\partial_t v + div(v \otimes v) + \nabla p = 0$$
$$div\ v = 0$$

*such that*

- $|v(x,t)| = 1$ *for a.e.* $(x,t) \in \Omega$
- $v(x,t) = 0$ *and* $p(x,t) = 0$ *for a.e.* $(x,t) \in \mathbb{R}^n_x \times \mathbb{R}_t \backslash \Omega$.

Instead of spelling out every details, which can be found in [11], we describe the geometric and functional analytic setup and outline the general strategy, specialized to the problem at hand. The main idea is consider the sets

$$K = \{(v,u) \in \mathbb{R}^n \times \mathcal{S}^n_0 : u = v \otimes v - \frac{1}{n}|v|^2 I_n,\ |v| = 1\}, \tag{3.6.24}$$

and

$$\mathcal{U} = int\ (conv\ K \times [-1,1]). \tag{3.6.25}$$

Clearly, a triple solving (3.1.14) (in the sense of the isomorphism) and taking values in the convex extreme points of $\overline{\mathcal{U}}$ is our desired solution. The plan is to prove $0 \in \mathcal{U}$, showing the existence of plane waves taking values in $\mathcal{U}$ and then add such plane waves to get an infinite sum

$$(v,u,q) = \sum_{i=1}^{\infty}(v_i,u_i,q_i) \tag{3.6.26}$$

with the properties that

- the partial sums $\sum_{i=1}^{\infty}(v_i,u_i,q_i)$ take values in $\mathcal{U}$,
- $(v,u,q)$ is supported in $\Omega$,
- $(v,u,q)$ takes values in the convex extreme points of $\overline{\mathcal{U}}$ a.e. in $\Omega$,
- $(v,u,q)$ solves the linear partial differential equations (3.1.14) (in the sense of the isomorphism).

One of the crucial facts that such a construction is possible is the largeness of the wave cone $\Lambda$. The key functional analytic set up is the following:
Let

$$X_0 := \{(v,u,q) \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}_t) : (1),(2),(3) \text{ below hold }\} \tag{3.6.27}$$

(1) $supp(v,u,q) \subset \Omega$,

(2) $(v,u,q)$ solves (3.1.14)(in the sense of the isomorphism) in $\mathbb{R}^n_x \times \mathbb{R}_t$

(3) $(v(x,t),u(x,t),q(x,t)) \in \mathcal{U}$ for all $(x,t) \in \mathbb{R}^n_x \times \mathbb{R}_t$.

We equip $X_0$ with the topology of $L^\infty$-weak-$*$ convergence of $(v,u,q)$ and we define $X$ to be the closure of $X_0$ in this topology. Now the set $X$ with this topology is a nonempty compact metrizable space. We fix a metric $d^*_\infty$ inducing the weak-$*$ topology of $L^\infty$ in $X$, so that $(X, d^*_\infty)$ is a complete metric space. Now, the identity maps

$$I : (X, d^*_\infty) \to L^2(\mathbb{R}^n_x \times \mathbb{R}_t) \text{ defined by } (v,u,q) \mapsto (v,u,q)$$

is a Baire-1 map and therefore the set of points of continuity is residual in $(X, d^*_\infty)$.

Proving that $0 \in \mathcal{U}$ is done by considering the linear map

$$T : C(S^{n-1}) \to \mathbb{R}^n \times \mathcal{S}_0^n,\ \phi \mapsto \int_{S^{n-1}} \left(v, v \otimes v - \frac{I_n}{n}\right) \phi(v) \mathrm{d}\mu,$$

where $\mu$ is the Haar measure on $S^{n-1}$. Clearly, if

$$\phi \geq 0 \text{ and } \int_{S^{n-1}} \phi \mathrm{d}\mu = 1,$$

then $T(\phi) \in conv\ K$. Also note that $T(1) = 0 = T(\alpha)$ and using $\alpha = 1 - \int_{S^{n-1}} \psi \mathrm{d}\mu$ and choosing a $\psi$ such that $\|\psi\|_{C(S^{n-1})} < \frac{1}{2}$, we obtain $T(B_{\frac{1}{2}}) \subset conv\ K$. Now we use intelligent choices of $\phi$ and the orthogonality in $L^2$ with a dimension counting argument to conclude $T$ is surjective.

Using the fact that $0 \in \mathcal{U}$ and the construction of localized plane waves, we can show that

**Lemma 3.6.8.** *There exists a dimensional constant $\beta > 0$ with the following property. Given $(v, u, q) \in X_0$ there exists a sequence $(v_k, u_k, q_k) \in X_0$ such that*

$$\|v_k\|^2_{L^2(\Omega)} \geq \|v\|^2_{L^2(\Omega)} + \beta \left(|\Omega| - \|v\|^2_{L^2(\Omega)}\right)^2, \tag{3.6.28}$$

*and*

$$(v_k, u_k, q_k) \overset{*}{\rightharpoonup} (v, u, q) \text{ in } L^\infty(\Omega). \tag{3.6.29}$$

Now if $(v, u, q)$ is a point of continuity of the identity map $I$, passing to the limit in (3.6.28) we obtain $|v| = 1$ a.e in $\Omega$, concluding the proof of Theorem (3.6.7).

# Chapter 4

# Conclusion

Differential inclusion techniques are not something new in mathematics literature. It has been used the context of optimal control for quite a long time, see for example [3],[1],[10], [9] , [30] among others. Also, in ordinary differential equation theory, Filippov's work, which led to e.g. defining generalized characteristics for many first order PDEs, Hamilton-Jacobi equation being one example, concerned with viewing an ODE as a differential inclusion.(Cf. [16]). Also notable is the work of Bressan and Flores ([2]). Differential geometry, or to be more precise, immersion-theoretic topology, gave birth to the method of convex integration. The works of Nash ([27], [28]) and Kuiper on imbedding problems ([22]) and Gromov's far reaching generalization of these works in [17], which gave convex integration method its modern form, is already quite old. The more recent works [15] and [31] and references therein proves that the volume of work done using these ideas in the context of nonlinear PDE is by no means small. So what exactly is so new and promising in the materials presented in this thesis?

Compensated compactness framework has been developed having in mind the equations arising out of questions in mathematical physics. On the other hand, the techniques in differential inclusions, though widely used in imbedding problems, i.e. problems arising out of questions in differential geometry because due to the underdetermined nature of the problem one expects high flexibility of solutions, was widely believed to be of no use in the context of PDEs that arises from mathematical physics, chiefly because *in questions of mathematical physics, generic uniqueness is to be expected, whereas the methods of differential inclusion provides nonuniqueness* . For example, Gromov's speech at the Balzan Prize clearly brings out the matter.

> *The class of infinitesimal laws subjugated by the homotopy principle is wide, but it does not include most partial differential equations (expressing infinitesimal laws ) of physics with a few exceptions in favour of this principle leading to unexpected solutions. In fact, the presence of h-principle would invalidate the very idea of a physical law as it yields very limited global information effected by the infinitesimal data.*

So it is precisely this new area of application, namely certain classes of PDEs arising in mathematical physics, of these tools that is new and a rather surprizing phenomenon. However, it is to be kept in mind that in the context of Calculus of variations, these ideas are being employed for quite some time now in the context of physically relevent questions of microstructure and phase transitions. But, the application of these ideas to evolution problems, especially hyperbolic systems can indeed turn out to be a breakthrough. On the other hand, even in PDEs of mathematical physics, nonuniqueness is not exactly new. The problem of nonuniqueness of the weak solution in PDEs of physics is usually referred to as the "selection criterion" problem. The choice of a proper selcetion criterion which singles out the physical solution from infinitely many weak solution is basically the question ( for such a question in the context of Euler equations, see [12]). But still untill recently, differential inclusion techniques have not been applied to PDEs of these types. The application of these methods to Euler equation can not only help our understanding of turbulence, but also it can open a new door towards a deeper understanding of nonlinear hyperbolic systems in general or transport equations. We may well hope that these fresh insurge of ideas into nonlinear PDEs of mathematical physics, especially transport equations and hyperbolic systems will deepen our understanding of mathematical description of nature.

# Appendix A

# Different Notions of Convexity

In this appendix we collect various facts related to different imporatnt notions of convexity in the vectorial calculus of variations. We define these notions of convexity and the corresponding hulls here and state some basic properties of them. More materials in these direction can be found in [6], [26] and [21].

## *A.1 Convexity for functions in the vectorial case*

For a matrix $F \in \mathcal{M}^{m\times n}$ let $\mathbf{M}(F)$ denote the vector that consists of all minors of $F$ and let $d(n,m) = \sum_{r=1}^{\min(n,m)} \binom{n}{r}\binom{m}{r}$ denote its length.

**Definition A.1.1.** A function $f : \mathcal{M}^{m\times n} \to \mathbb{R} \cup \{+\infty\}$ is

(i) *convex* if

$$f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda) f(B)$$
$$\forall A, B \in \mathcal{M}^{m\times n},\ \lambda \in (0,1);$$

(ii) *polyconvex* if there exists a convex function $g : \mathbb{R}^{d(n,m)} \to \mathbb{R} \cup \{+\infty\}$ such that

$$f(F) = g(\mathbf{M}(F));$$

(iii) *quasiconvex* if for every open and bounded set $\Omega$ with $meas(\partial\Omega) = 0$ one has

$$\int_\Omega (f(F + \nabla\phi) - f(F))\mathrm{d}x \geq 0 \quad \forall \phi \in W_0^{1,\infty}(\Omega; \mathbb{R}^m),$$

whenever the integral on the left side exists;

(iv) *rank-*1 *convex*, if $f$ is convex along rank-1 lines, i.e. if

$$f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda) f(B)$$
$$\forall A, B \in \mathcal{M}^{m\times n} \text{ with } rank(B-A) = 1,\ \lambda \in (0,1);$$

Now we record a theorem regarding the interelations of these notions of convexity.

**Theorem A.1.2.** *(i) Let $f : \mathcal{M}^{m\times n} \to \mathbb{R}$, then,*

$$f \text{ convex} \Rightarrow f \text{ polyconvex} \Rightarrow f \text{ quasiconvex} \Rightarrow f \text{ rank one convex.}$$

*If $f : \mathcal{M}^{m\times n} \to \mathbb{R}\cup\{+\infty\}$, then*

$$f \text{ convex } \Rightarrow f \text{ polyconvex } \Rightarrow f \text{ rank one convex.}$$

*(ii) If $m = 1$ or $n = 1$, then all these notions are equivalent.*

*(iii) If $f : \mathcal{M}^{m\times n} \to \mathbb{R}$ is convex, polyconvex, quasiconvex, rank one convex, then $f$ is locally Lipschitz.*

Another notion for convexity is usually called $\mathcal{D}$-convexity.

**Definition A.1.3.** Given a set $\mathcal{D} \subset \mathcal{M}^{m\times n}$, we call a function $f : \mathcal{M}^{m\times n} \to \mathbb{R}$ is $\mathcal{D}$-convex if it is convex along directions in $\mathcal{D}$, i.e.

$$f(\lambda A + (1-\lambda)B) \le \lambda f(A) + (1-\lambda)f(B)$$
$$\forall A, B \in \mathcal{M}^{m\times n} \text{ with } B - A \in \mathcal{D},\ \lambda \in (0,1);$$

Note that when $\mathcal{D}$ is the set of all rank one $m \times n$ matrices, $\mathcal{D}$-convexity is just rank one convexity.

## *A.2 Convex envelopes of functions*

We define the *convex*, *polyconvex*, *quasiconvex* and *rank one convex envelopes* respectively as,

$$Cf = \sup\{g \le f : g \text{ convex}\} \tag{A.2.1}$$
$$Pf = \sup\{g \le f : g \text{ polyconvex}\} \tag{A.2.2}$$
$$Qf = \sup\{g \le f : g \text{ quasiconvex}\} \tag{A.2.3}$$
$$Rf = \sup\{g \le f : g \text{ rank one convex}\}. \tag{A.2.4}$$

**Theorem A.2.1.** *We have*

$$Cf \le Pf \le Qf \le Rf \le f. \tag{A.2.5}$$

*Also, if $n = 1$ or $m = 1$, then $Cf = Pf = Qf = Rf$.*

## *A.3 Convex hulls for sets*

For a compact set $K$ we define the rank-one convex,quasiconvex and polyconvex hull as the set of those points which can not be seperated by the corresponding class of functions.

**Definition A.3.1.** For a compact set $K$, we define

$$K^* := \{A \in \mathcal{M}^{m\times n} : f(A) \le \sup_K f,\ f \text{ is } *\}, \quad * = \{c, rc, qc, pc\}. \tag{A.3.1}$$

where c, rc, qc, pc stand for convex, rank-one convex, quasiconvex and polyconvex respectively and $K^c$, $K^{rc}$, $K^{qc}$, $K^{pc}$ denotes the corresponding hulls of K. For a general set $E$, we set

$$E^* = \bigcup_{K\subset E \text{ compact}} K^*, \quad * = \{rc, qc, pc\}. \tag{A.3.2}$$

Of course, we say a set $E$ is *polyconvex*, *quasiconvex*, *rank one convex* if $E = E^*$, where $*$ denotes pc, qc, rc respectively.

**Definition A.3.2.** A set $K$ is called *lamination convex* if the conditions $A, B \in K$ and $rank(B - A) = 1$ imply that convex combinations of $A$ and $B$ are in $K$.

The lamination convex hull $K^{lc}$ is the smallest lamination convex set containing $K$, which can equivalently be defined in an inductive way by the following:

$$K^{lc} = \cup_{i=0}^{\infty} K^{(i)}, where\ K^{(0)} = K,$$
$$K^{(i+1)} = K^{(i)} \cup \{[A, B] : A, B \in K^{(i)},\ rank(B - A) = 1\}.$$

As for their interelations, we have

$$K^{lc} \subset K^{rc} \subset K^{qc} \subset K^{pc} \subset K^{c}. \tag{A.3.3}$$

## *A.4 Dual objects: laminates and gradient Young measures*

Let $\mathcal{P}(K)$ denote the set of probability measures supported on $K$. For $\mu \in \mathcal{P}(K)$ we denote by $\bar{\mu} = \int A \mathrm{d}\mu(A)$ its barycentre. We consider the following subsets of $\mathcal{P}(K)$ which satisfy a Jensen's inequality with respect to the convexity notions described above.

$$\mathcal{M}^{*}(K) = \{\mu \in \mathcal{P}(K) : \int f(A)\mathrm{d}\mu(A) \geq f(\bar{\mu}),\ \forall f\ *\}. \tag{A.4.1}$$

where $* = \{rc, qc, pc\}$.
For a compact set $K$ we have

$$K^{*} = \{\bar{\mu} : \mu \in \mathcal{M}^{*}(K)\}, \quad * = \{rc, qc, pc\}. \tag{A.4.2}$$

The elements of $\mathcal{M}^{qc}(\mathbb{R}^{m\times n})$ are called homogeneous gradient Young measures and the elements of $\mathcal{M}^{rc}(K)$ are called laminates.

# Bibliography


[1] Jean-Pierre Aubin and Arrigo Cellina. *Differential inclusions*, volume 264 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1984. Set-valued maps and viability theory.

[2] Alberto Bressan and Fabián Flores. On total differential inclusions. *Rend. Sem. Mat. Univ. Padova*, 92:9–16, 1994.

[3] Arrigo Cellina. On the differential inclusion $x' \in [-1, +1]$. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8)*, 69(1-2):1–6 (1981), 1980.

[4] Gui-Qiang Chen. Compactness methods and nonlinear hyperbolic conservation laws. In *Some current topics on nonlinear conservation laws*, volume 15 of *AMS/IP Stud. Adv. Math.*, pages 33–75. Amer. Math. Soc., Providence, RI, 2000.

[5] Bernard Dacorogna. *Weak continuity and weak lower semicontinuity of nonlinear functionals*, volume 922 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1982.

[6] Bernard Dacorogna. *Direct methods in the calculus of variations*, volume 78 of *Applied Mathematical Sciences*. Springer-Verlag, Berlin, 1989.

[7] Bernard Dacorogna and Paolo Marcellini. General existence theorems for Hamilton-Jacobi equations in the scalar and vectorial cases. *Acta Math.*, 178(1):1–37, 1997.

[8] Bernard Dacorogna and Paolo Marcellini. *Implicit partial differential equations*. Progress in Nonlinear Differential Equations and their Applications, 37. Birkhäuser Boston Inc., Boston, MA, 1999.

[9] F. S. De Blasi and G. Pianigiani. A Baire category approach to the existence of solutions of multivalued differential equations in Banach spaces. *Funkcial. Ekvac.*, 25(2):153–162, 1982.

[10] F. S. De Blasi and G. Pianigiani. Non-convex-valued differential inclusions in Banach spaces. *J. Math. Anal. Appl.*, 157(2):469–494, 1991.

[11] Camillo De Lellis and Jr. László Székelyhidi. The Euler equations as a differential inclusion. *Ann. of Math. (2)*, 170(3):1417–1436, 2009.

[12] Camillo De Lellis and Jr. László Székelyhidi. On admissibility criteria for weak solutions of the Euler equations. *Arch. Ration. Mech. Anal.*, 195(1):225–260, 2010.

[13] Camillo De Lellis and Jr. László Székelyhidi. The $h$-principle and the equations of fluid dynamics. *arXiv:1111.2700v1*, 2011.

[14] Ronald J. DiPerna. Compensated compactness and general systems of conservation laws. *Trans. Amer. Math. Soc.*, 292(2):383–420, 1985.

[15] Y. Eliashberg and N. Mishachev. *Introduction to the h-principle*, volume 48 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002.

[16] A. F. Filippov. *Differential equations with discontinuous righthand sides*, volume 18 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1988. Translated from the Russian.

[17] Mikhael Gromov. *Partial differential relations*, volume 9 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1986.

[18] J.-L. Joly, G. Métivier, and J. Rauch. Trilinear compensated compactness and nonlinear geometric optics. *Ann. of Math. (2)*, 142(1):121–169, 1995.

[19] Bernd Kirchheim. Deformations with finitely many gradients and stability of quasiconvex hulls. *C. R. Acad. Sci. Paris Sér. I Math.*, 332(3):289–294, 2001.

[20] Bernd Kirchheim. Rigidity and geometry of microstructures. *Habilitation thesis*, 2003.

[21] Bernd Kirchheim, Stefan Müller, and Vladimír Šverák. Studying nonlinear pde by geometry in matrix space. In *Geometric analysis and nonlinear partial differential equations*, pages 347–395. Springer, Berlin, 2003.

[22] Nicolaas H. Kuiper. On $C^1$-isometric imbeddings. I, II. *Nederl. Akad. Wetensch. Proc. Ser. A.* **58** *= Indag. Math.*, 17:545–556, 683–689, 1955.

[23] Yunguang Lu. *Hyperbolic conservation laws and the compensated compactness method*, volume 128 of *Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics*. Chapman & Hall/CRC, Boca Raton, FL, 2003.

[24] S. Müller and M. A. Sychev. Optimal existence theorems for nonhomogeneous differential inclusions. *J. Funct. Anal.*, 181(2):447–475, 2001.

[25] S. Müller and V. Šverák. Convex integration for Lipschitz mappings and counterexamples to regularity. *Ann. of Math. (2)*, 157(3):715–742, 2003.

[26] Stefan Müller. Variational models for microstructure and phase transitions. In *Calculus of variations and geometric evolution problems (Cetraro, 1996)*, volume 1713 of *Lecture Notes in Math.*, pages 85–210. Springer, Berlin, 1999.

[27] John Nash. $C^1$ isometric imbeddings. *Ann. of Math. (2)*, 60:383–396, 1954.

[28] John Nash. The imbedding problem for Riemannian manifolds. *Ann. of Math. (2)*, 63:20–63, 1956.

[29] Pablo Pedregal. *Parametrized measures and variational principles.* Progress in Nonlinear Differential Equations and their Applications, 30. Birkhäuser Verlag, Basel, 1997.

[30] Giulio Pianigiani. Differential inclusions. The Baire category method. In *Methods of nonconvex analysis (Varenna, 1989)*, volume 1446 of *Lecture Notes in Math.*, pages 104–136. Springer, Berlin, 1990.

[31] David Spring. *Convex integration theory*, volume 92 of *Monographs in Mathematics.* Birkhäuser Verlag, Basel, 1998. Solutions to the $h$-principle in geometry and topology.

[32] M. A. Sychev. Comparing two methods of resolving homogeneous differential inclusions. *Calc. Var. Partial Differential Equations*, 13(2):213–229, 2001.

[33] M. A. Sychev. A few remarks on differential inclusions. *Proc. Roy. Soc. Edinburgh Sect. A*, 136(3):649–668, 2006.

[34] L. Tartar. Compensated compactness and applications to partial differential equations. In *Nonlinear analysis and mechanics: Heriot-Watt Symposium, Vol. IV*, volume 39 of *Res. Notes in Math.*, pages 136–212. Pitman, Boston, Mass., 1979.

[35] Luc Tartar. The compensated compactness method applied to systems of conservation laws. In *Systems of nonlinear partial differential equations (Oxford, 1982)*, volume 111 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 263–285. Reidel, Dordrecht, 1983.

[36] Luc Tartar. *The general theory of homogenization*, volume 7 of *Lecture Notes of the Unione Matematica Italiana*. Springer-Verlag, Berlin, 2009. A personalized introduction.