Differential Inclusion approach in Nonlinear PDEs

Swarnendu Sil

M.Phil Thesis Talk
Thesis Advisor: Prof. M. Vanninathan

Tata Institute of Fundamental Research
Centre for Applicable Mathematics
Bangalore, India

20th July, 2012

Introduction

In this talk, **oscillations in nonlinear partial differential equations** is, in a sense, our main theme.

Introduction

In this talk, **oscillations in nonlinear partial differential equations** is, in a sense, our main theme.

More particularly, the framework of **compensated compactness**, introduced by Tartar (see [Tartar, 1979]), to analyze oscillations in sequences of approximate or exact solutions to nonlinear partial differential equations (henceforth PDEs) and its connection and interrelation with **methods to solve differential inclusions** and the application of these ideas in **nonlinear PDEs**.

Introduction

In this talk, **oscillations in nonlinear partial differential equations** is, in a sense, our main theme.

More particularly, the framework of **compensated compactness**, introduced by Tartar (see [Tartar, 1979]), to analyze oscillations in sequences of approximate or exact solutions to nonlinear partial differential equations (henceforth PDEs) and its connection and interrelation with **methods to solve differential inclusions** and the application of these ideas in **nonlinear PDEs**.

So our main focus will be on

- **Compensated compactness**
- **Differential Inclusions**
- **Casting nonlinear PDEs as differential inclusions and using the above framework to construct solutions**

Casting nonlinear PDEs in the framework

Many nonlinear partial differential equations can be written as a **system of linear PDEs** together with a **nonlinear pointwise constitutive relations**.

Casting nonlinear PDEs in the framework

Many nonlinear partial differential equations can be written as a **system of linear PDEs** together with a **nonlinear pointwise constitutive relations**.

Motivation

The motivation for the idea comes from physics, since most PDEs in physics are indeed derived using this very principle in the first place. One figures out the **constitutive relations** directly from the physical law and one has the **balance equations** for certain quantities. One then substitutes the constitutive relations into the balance equations and obtain the final form of the PDE. Since most equations in physics are derived in this way in the first place, there can be little doubt that these nonlinear PDEs can obviously be cast into that form.

Framework

Idea

We want to disentangle the role of the **linear differential structure** and the **algebraic nonlinear structure** of the **nonlinear operator**.

Framework

Idea

We want to disentangle the role of the **linear differential structure** and the **algebraic nonlinear structure** of the **nonlinear operator**.

Setting

We consider nonlinear PDEs that can be expressed as a system of linear PDEs, called **(balance laws)**

$$\sum_{i=1}^{m} A_i \partial_i z = 0 \tag{1}$$

coupled with a pointwise nonlinear constraint **(constitutive relations)**

$$z(x) \in K \subset \mathbb{R}^d \quad a.e.\ x \in \Omega \tag{2}$$

where $z : \Omega \subset \mathbb{R}^m \to \mathbb{R}^d$ is the unknown state variable.

General Problem

The general problem of the compensated compactness framework can now be stated as follows:

General Problem

The general problem of the compensated compactness framework can now be stated as follows: Describe the oscillations in a **weakly convergent sequence of functions** $z^{\epsilon} : \Omega \subset \mathbb{R}^m \to \mathbb{R}^d$ subject to linear differential constraints, called *balance laws*, of the form

$$\sum_{i=1}^{m} A_i \partial_i z^{\epsilon} = \phi^{\epsilon} \tag{3}$$

and nonlinear algebraic constraints, called *constitutive relations*, of the form

$$\{z^{\epsilon}(y)\} \subset M \tag{4}$$

for almost all y in Ω.

General Problem

The general problem of the compensated compactness framework can now be stated as follows: Describe the oscillations in a **weakly convergent sequence of functions** $z^{\epsilon} : \Omega \subset \mathbb{R}^m \to \mathbb{R}^d$ subject to linear differential constraints, called *balance laws*, of the form

$$\sum_{i=1}^{m} A_i \partial_i z^{\epsilon} = \phi^{\epsilon} \tag{3}$$

and nonlinear algebraic constraints, called *constitutive relations*, of the form

$$\{z^{\epsilon}(y)\} \subset M \tag{4}$$

for almost all y in Ω. Here A_i denotes a constant $s \times d$ matrix with s arbitrary and fixed. M is a subset of the *state space* $\mathbb{R}^d$ and is usually a manifold.

General Problem

The general problem of the compensated compactness framework can now be stated as follows: Describe the oscillations in a **weakly convergent sequence of functions** $z^\epsilon : \Omega \subset \mathbb{R}^m \to \mathbb{R}^d$ subject to linear differential constraints, called *balance laws*, of the form

$$\sum_{i=1}^{m} A_i \partial_i z^\epsilon = \phi^\epsilon \quad (3)$$

and nonlinear algebraic constraints, called *constitutive relations*, of the form

$$\{z^\epsilon(y)\} \subset M \quad (4)$$

for almost all y in Ω. Here A_i denotes a constant $s \times d$ matrix with s arbitrary and fixed. M is a subset of the *state space* $\mathbb{R}^d$ and is usually a manifold.

In compensated compactness theory, one is particularly interested in determining how differential and algebraic constraints collaborate to suppress oscillations in z^ϵ.

Our Target

We do not want to ascertain that the differential structure **kills all oscillations** that the nonlinearity can not suppress.

Our Target

We do not want to ascertain that the differential structure **kills all oscillations** that the nonlinearity can not suppress.
Instead we want the differential structure to **allow many types of oscillation**, so that the balance laws (1) has a large, infinite family of oscillatory solutions.

Our Target

We do not want to ascertain that the differential structure **kills all oscillations** that the nonlinearity can not suppress.
Instead we want the differential structure to **allow many types of oscillation**, so that the balance laws (1) has a large, infinite family of oscillatory solutions.
We want this family to be so large that

Our Target

We do not want to ascertain that the differential structure **kills all oscillations** that the nonlinearity can not suppress.
Instead we want the differential structure to **allow many types of oscillation**, so that the balance laws (1) has a large, infinite family of oscillatory solutions.
We want this family to be so large that

- **It is possible to ensure that there are rich supply of solutions to (1), which takes values in some convex hulls of K,**

Our Target

We do not want to ascertain that the differential structure **kills all oscillations** that the nonlinearity can not suppress.

Instead we want the differential structure to **allow many types of oscillation**, so that the balance laws (1) has a large, infinite family of oscillatory solutions.

We want this family to be so large that

- **It is possible to ensure that there are rich supply of solutions to (1), which takes values in some convex hulls of K,**
- **We can perturb these solutions of (1) by adding controlled oscillation to generate a sequence in such a way that each term of the sequence still remains a solution of (1), but the image of each term approaches the constitutive set K.**

Our Target

We do not want to ascertain that the differential structure **kills all oscillations** that the nonlinearity can not suppress.
Instead we want the differential structure to **allow many types of oscillation**, so that the balance laws (1) has a large, infinite family of oscillatory solutions.
We want this family to be so large that

- **It is possible to ensure that there are rich supply of solutions to (1), which takes values in some convex hulls of K,**
- **We can perturb these solutions of (1) by adding controlled oscillation to generate a sequence in such a way that each term of the sequence still remains a solution of (1), but the image of each term approaches the constitutive set K.**

Moreover, we want to be able to add controlled oscillations allowed by the differential structure in such a way such that the resulting sequence of solutions to (1), **apriori only weakly convergent**, actually converges strongly and the strong limit of this sequence takes its values in K.

The strong convergence is possible precisely because of **our ability to add controlled oscillation** and is proved either by *convex integration* or *Baire category* arguments, two already familiar techniques in the theory of **differential inclusions**.

The strong convergence is possible precisely because of **our ability to add controlled oscillation** and is proved either by *convex integration* or *Baire category* arguments, two already familiar techniques in the theory of **differential inclusions**.

The theory of differential inclusions have a long history though their application to nonlinear PDEs are rather new. Cellina considered ordinary differential inclusions related to problems in optimal control (Cf. [Cellina, 1980], [Aubin and Cellina, 1984]). Gromov studied extensively partial differential inclusions in his classic 'Partial differential relations' [Gromov, 1986]. An excellent reference for both differential inclusions and their applications to nonlinear PDEs can be found in [Dacorogna and Marcellini, 1999], [Dacorogna and Marcellini, 1997]. We shall illustrate the basic idea of solving a differential inclusion by using the most well-studied example of partial differential inclusions, namely the **gradient inclusion**. We will also show how these ideas together with compensated compactness framework gives rise to a mechanism for generating oscillatory solutions to nonlinear PDEs.

General Strategy

- Cast nonlinear PDEs in the framework of Compensated Compactness.

General Strategy

- Cast nonlinear PDEs in the framework of Compensated Compactness.
- Reduce it to a differential inclusion.

General Strategy

- Cast nonlinear PDEs in the framework of Compensated Compactness.
- Reduce it to a differential inclusion.
- Solve the inclusion.

General Strategy

- Cast nonlinear PDEs in the framework of Compensated Compactness.
- Reduce it to a differential inclusion.
- Solve the inclusion.

We solve the inclusion by

General Strategy

- Cast nonlinear PDEs in the framework of Compensated Compactness.
- Reduce it to a differential inclusion.
- Solve the inclusion.

We solve the inclusion by

- Understanding the possible oscillations of sequences of approximate solutions

General Strategy

- Cast nonlinear PDEs in the framework of Compensated Compactness.
- Reduce it to a differential inclusion.
- Solve the inclusion.

We solve the inclusion by

- Understanding the possible oscillations of sequences of approximate solutions
- Finding a way to add controlled oscillation to a sequence of approximate solutions

General Strategy

- Cast nonlinear PDEs in the framework of Compensated Compactness.
- Reduce it to a differential inclusion.
- Solve the inclusion.

We solve the inclusion by

- Understanding the possible oscillations of sequences of approximate solutions
- Finding a way to add controlled oscillation to a sequence of approximate solutions
- Obtaining strong convergence by iterating this procedure.

General Strategy

- Cast nonlinear PDEs in the framework of Compensated Compactness.
- Reduce it to a differential inclusion.
- Solve the inclusion.

We solve the inclusion by

- Understanding the possible oscillations of sequences of approximate solutions
- Finding a way to add controlled oscillation to a sequence of approximate solutions
- Obtaining strong convergence by iterating this procedure.
- Passing to the limit, that is, by showing the strong limit solves the PDE

Compensated Compactness

Let $z_n : \Omega \subset \mathbb{R}^m \to \mathbb{R}^d$ be a sequence of functions such that $z_n \rightharpoonup z$ in $(L^\infty(\Omega))^d$ weak $*$. Also $\{z_n\}$ satisfies a system of linear differential constraints, called *balance laws*, of the form

$$\sum_{i=1}^{m} A_i \partial_i z_n = \phi_n \tag{5}$$

and nonlinear algebraic constraints, called *constitutive relations*, of the form

$$z_n(x) \in K \tag{6}$$

for all n for almost all x in Ω. Here A_i denotes a constant $s \times d$ matrix with s arbitrary and fixed. Ω is a bounded open subset of $\mathbb{R}^m$ and K is a subset of the *state space* $\mathbb{R}^d$. Assume also that $\phi_n \to \phi$ in some suitable topology on a suitable function space.

We want to determine when it is possible to assert that

$$\sum_{i=1}^{m} A_i \partial_i z = \phi \tag{7}$$

and

$$z(x) \in K \tag{8}$$

for a.e. x in Ω.

We want to determine when it is possible to assert that

$$\sum_{i=1}^{m} A_i \partial_i z = \phi \tag{7}$$

and

$$z(x) \in K \tag{8}$$

for a.e. x in Ω.

Note that these questions are not the ones we will be pursuing for long. But we are more interested in the framework than the theory. Understanding the effect of linear differential structure and the nonlinear algebraic structure seperately on the possible oscillations of a weakly convergent sequence of approximate solutions is our aim. The principle aim of this talk is show that this knowledge might also helps us to solve problems which are in a sense, outside what was initially thought to be the target of compensated compactness theory.

Without Differential Constraints

The first question we can ask is whether we can assert that $z(y) \in K$ for a.e y in Ω without any information on the derivatives of z_n, that is, without (5). More precisely, we are asking if the informations $z_n \rightharpoonup z$ in $(L^\infty(\Omega))^d$ weak $*$ and $z_n(x) \in K$ for a.e. x in Ω is enough to imply $z(y) \in K$ for a.e y in Ω. The following theorem answers this question in the negative.

Without Differential Constraints

The first question we can ask is whether we can assert that $z(y) \in K$ for a.e y in Ω without any information on the derivatives of z_n, that is, without (5). More precisely, we are asking if the informations $z_n \rightharpoonup z$ in $(L^\infty(\Omega))^d$ weak $*$ and $z_n(x) \in K$ for a.e. x in Ω is enough to imply $z(y) \in K$ for a.e y in Ω. The following theorem answers this question in the negative.

Theorem

(1) Let $z_n : \Omega \to \mathbb{R}^d$ be such that $z_n \rightharpoonup z$ in $(L^\infty(\Omega))^d$ weak $*$ and $z_n(x) \in K$ a.e. Then $z(x) \in \overline{conv}(K)$ a.e (where $\overline{conv}(K)$ is the closed convex hull of K).
(2) Conversely, let $z \in (L^\infty(\Omega))^d$ and $z(x) \in \overline{conv}(K)$ a.e, then there exists a sequence $\{z_n\}$ such that $z_n \rightharpoonup z$ in $(L^\infty(\Omega))^d$ weak $*$ and $z_n(x) \in K$ a.e. for all n.

The following lemma is crucial in the proof of the theorem..

Lemma

If f is a convex continuous function on $\mathbb{R}^d$, if $z_n \rightharpoonup z$ in $(L^\infty(\Omega))^d$ weak $*$, and if $f(z_n) \rightharpoonup l$ then $l \geq f(z)$ a.e.

Lemma 2 immediately suggests the following questions:

Q1. For which functions f do we have $f(z_n) \rightharpoonup f(z)$, that is, f is sequentially weakly continuous ?

Q2. For which functions f do we have if $f(z_n) \rightharpoonup l$, then $l \geq f(z)$, that is, f is sequentially weakly lower semicontinuous?

Lemma 2 immediately suggests the following questions:

Q1. For which functions f do we have $f(z_n) \rightharpoonup f(z)$, that is, f is sequentially weakly continuous ?

Q2. For which functions f do we have if $f(z_n) \rightharpoonup l$, then $l \geq f(z)$, that is, f is sequentially weakly lower semicontinuous?

Theorem

Let $z_n : \Omega \to \mathbb{R}^d$ be such that $z_n \rightharpoonup z$ in $(L^\infty(\Omega))^d$ weak $*$ and $z_n(x) \in K$ a.e.. Then we have,

(1) *$z(x) \in K$ if and only if K is closed and convex.*

(2) *$f(z_n) \rightharpoonup f(z)$ in $(L^\infty)^N$ weak $*$ if and only if f is affine on $\overline{conv}(K)$.*

(3) *If $f(z_n) \rightharpoonup l$ in $(L^\infty)^N$ weak $*$, then $l \geq f(z)$ if and only if f is convex on $\overline{conv}(K)$.*

With Differential Constraints

We state the problem in the same way as before.Ω and K are both assumed bounded.

$$u_1^\epsilon, \ldots, u_d^\epsilon \rightharpoonup u_1, \ldots, u_d \text{ in } L^\infty(\Omega) \text{ weak } * \tag{9}$$

$$u^\epsilon \in K \text{ a.e.} \tag{10}$$

$$\sum_{j,k} a_{ijk} \frac{\partial u_j^\epsilon}{\partial x_k} \in \text{ bounded set in } L^\infty \tag{11}$$

for $i = 1, \ldots, q$ where the a_{ijk} are real constants.

With Differential Constraints

We state the problem in the same way as before.Ω and K are both assumed bounded.

$$u_1^\epsilon, \ldots, u_d^\epsilon \rightharpoonup u_1, \ldots, u_d \text{ in } L^\infty(\Omega) \text{ weak } * \tag{9}$$

$$u^\epsilon \in K \text{ a.e.} \tag{10}$$

$$\sum_{j,k} a_{ijk} \frac{\partial u_j^\epsilon}{\partial x_k} \in \text{ bounded set in } L^\infty \tag{11}$$

for $i = 1, \ldots, q$ where the a_{ijk} are real constants.

The questions are the same as before, that is,

Q1. Does $u(x) \in K$ a.e.?

Q2. For which functions f do we have $f(u^\epsilon) \rightharpoonup f(u)$, that is, f is sequentially weakly continuous ?

Q3. For which functions f do we have if $f(u^\epsilon) \rightharpoonup l$, then $l \geq f(u)$, that is, f is sequentially weakly lower semicontinuous?

The partial answer to these questions will rely heavily on the following definitions:

Definition (Oscillation Variety:)

The oscillation variety corresponding to (11) is the set $\mathcal{V}$, defined by,

$$\mathcal{V} := \left\{ (\lambda, \xi) \in \mathbb{R}^d \times \mathbb{R}^m \setminus \{0\} : \sum_{j,k} a_{ijk} \lambda_j \xi_k = 0 \quad \textit{for } i = 1, \ldots, q \right\}. \qquad (12)$$

The partial answer to these questions will rely heavily on the following definitions:

Definition (Oscillation Variety:)

The oscillation variety corresponding to (11) is the set $\mathcal{V}$, defined by,

$$\mathcal{V} := \left\{ (\lambda, \xi) \in \mathbb{R}^d \times \mathbb{R}^m \setminus \{0\} : \sum_{j,k} a_{ijk} \lambda_j \xi_k = 0 \quad for \ i = 1, \ldots, q \right\}. \tag{12}$$

Definition (Wave Cone:)

The wave cone corresponding to (11) is the projection of $\mathcal{V}$ on the physical space, defined by,

$$\Lambda := \left\{ \lambda \in \mathbb{R}^d : \exists \xi \in \mathbb{R}^m \setminus \{0\} \quad with \ \sum_{j,k} a_{ijk} \lambda_j \xi_k = 0 \quad for \ i = 1, \ldots, q \right\}. \tag{13}$$

Remark

(1) *In the case without derivatives, we have* $\mathcal{V} = \mathbb{R}^d \times \mathbb{R}^m \backslash \{0\}$ *and* $\Lambda = \mathbb{R}^d$*. If* $\Lambda = \mathbb{R}^d$*, this means (11) contains very little information.*

(2) *In the compactness case, we have* $\Lambda = \{0\}$*, and this happens if the list (11) contains all the derivatives* $\dfrac{\partial u_j^\epsilon}{\partial x_k}$ *seperately , which are thus all bounded.*

Remark

(1) *In the case without derivatives, we have* $\mathcal{V} = \mathbb{R}^d \times \mathbb{R}^m \backslash \{0\}$ *and* $\Lambda = \mathbb{R}^d$. *If* $\Lambda = \mathbb{R}^d$, *this means (11) contains very little information.*

(2) *In the compactness case, we have* $\Lambda = \{0\}$, *and this happens if the list (11) contains all the derivatives* $\dfrac{\partial u_j^\epsilon}{\partial x_k}$ *seperately , which are thus all bounded.*

Theorem

If (9), (10) and (11) imply $u(x) \in K$ *a.e., then* K *must satisfy the following conditions:*

NC0: K *is closed.*

NC1: *If* $a, b \in \overline{K} = K$ *and* $b - a \in \Lambda$, *then the segment* $[a, b]$ *is in* K.

Corollary

A necessary condition on K and f so that (Q2) can be answered positively is:
If $a_1, a_2, \ldots, a_d \in K,\ \xi \neq 0$ with $(a_j - a_k, \xi) \in \mathcal{V}$ for all j, k, then f is affine on $conv\{a_1, \ldots, a_d\}$.

Corollary

A necessary condition on K and f so that (Q2) can be answered positively is: If $a_1, a_2, \ldots, a_d \in K$, $\xi \neq 0$ with $(a_j - a_k, \xi) \in \mathcal{V}$ for all j, k, then f is affine on $conv\{a_1, \ldots, a_d\}$.

Corollary

A necessary condition on K and f such that (Q3) can be answered positively is: If $a_1, a_2, \ldots, a_d \in K$, $\xi \neq 0$ with $(a_j - a_k, \xi) \in \mathcal{V}$ for all j, k, then f is convex on $conv\{a_1, \ldots, a_d\}$.

Theorem

Suppose we have the following hypotheses

$(H1')$ $u_i^\epsilon \rightharpoonup u_i$ *in* $L^2(\Omega)$ *weak for* $i = 1, \ldots, d$

$(H3')$ $\sum_{j,k} a_{ijk} \frac{\partial u_j^\epsilon}{\partial x_k} \in$ *a compact set (for the strong topology) of* $H^{-1}_{loc}(\Omega)$,
for $i = 1, \ldots, d$.

Then if Q *is quadratic and satisfies* $Q(\lambda) \geq 0$ *for all* $\lambda \in \Lambda$, *and if* $Q(u^\epsilon) \rightharpoonup l$ *in the sense of distributions (* l *may be a measure), then*

$$l \geq Q(u) \quad \textit{(in the sense of measures).}$$

Remark

(1) *If* Q *is quadratic to say that* Q *is* Λ*-convex is equivalent to saying* $Q(\lambda) \geq 0$, *for all* $\lambda \in \Lambda$.

(2) *This theorem asserts that if* f *is quadratic then the necessary condition stated above in Corollary is also sufficient.*

Corollary

If Q is quadratic and satisfies $Q(\lambda) = 0$ for all $\lambda \in \Lambda$, and if $\{u^\epsilon\}$ satisfies $(H1')$ and $(H3')$, then $Q(u^\epsilon) \rightharpoonup Q(u)$ in the sense of distributions.

Corollary

If Q is quadratic and satisfies $Q(\lambda) = 0$ for all $\lambda \in \Lambda$, and if $\{u^\epsilon\}$ satisfies $(H1')$ and $(H3')$, then $Q(u^\epsilon) \rightharpoonup Q(u)$ in the sense of distributions.

Corollary

Div-Curl Lemma: Let

$$\begin{aligned}
v^\epsilon &\rightharpoonup v && \textit{in } (L^2(\Omega))^m \textit{ weak }, && (14)\\
w^\epsilon &\rightharpoonup w && \textit{in } (L^2(\Omega))^m \textit{ weak }, && (15)\\
div\ v^\epsilon &\to div\ v && \textit{in } H^{-1}(\Omega) \textit{ strong }, && (16)\\
curl\ w^\epsilon &\to curl\ w && \textit{in } (H^{-1}(\Omega))^{m^2} \textit{ strong}. && (17)
\end{aligned}$$

Then

$$v^\epsilon \cdot w^\epsilon \rightharpoonup v \cdot w \qquad \textit{in } \mathcal{D}'(\Omega) \tag{18}$$

and this is the only nonlinear functional that is (sequentially) weakly continuous.

Upto this point we asked the question:

Upto this point we asked the question:

Given a sequence of approximate solutions $\{z_n\}$ satisfying (5) and (6), when is it possible to assert that the weak limit z of the sequence $\{z_n\}$ satisfies (7) and (8).

Upto this point we asked the question:

Given a sequence of approximate solutions $\{z_n\}$ satisfying (5) and (6), when is it possible to assert that the weak limit z of the sequence $\{z_n\}$ satisfies (7) and (8).

Now we will ask a question which is, in a sense reverse of the previous one.

Given a sequence of approximate solutions $\{z_n\}$ satisfying

$$\sum_{i=1}^{m} A_i \partial_i z_n = \phi \tag{19}$$

and

$$z_n(x) \in K_n \tag{20}$$

for all n for almost all x and K_n is a subset of the *state space* $\mathbb{R}^d$ containing K for all n, when can we assert that there exists a z satisfying (7) and (8).

Upto this point we asked the question:

Given a sequence of approximate solutions $\{z_n\}$ satisfying (5) and (6), when is it possible to assert that the weak limit z of the sequence $\{z_n\}$ satisfies (7) and (8).

Now we will ask a question which is, in a sense reverse of the previous one.

Given a sequence of approximate solutions $\{z_n\}$ satisfying

$$\sum_{i=1}^{m} A_i \partial_i z_n = \phi \tag{19}$$

and

$$z_n(x) \in K_n \tag{20}$$

for all n for almost all x and K_n is a subset of the *state space* $\mathbb{R}^d$ containing K for all n, when can we assert that there exists a z satisfying (7) and (8).

This question however is a hopeless pursuit. **Too little information to conclude anything meaningful**.

With a lot of hindsight we rephrase and refine the question

if we know that there is set $\mathcal{U}$ such that it is possible to find solutions of (19) taking values in $\mathcal{U}$ and it is possible to push the distribution of its values inside $\mathcal{U}$ without ever leaving the solution set of (19), can we solve (19) and (20) ?

With a lot of hindsight we rephrase and refine the question

if we know that there is set $\mathcal{U}$ such that it is possible to find solutions of (19) taking values in $\mathcal{U}$ and it is possible to push the distribution of its values inside $\mathcal{U}$ without ever leaving the solution set of (19), can we solve (19) and (20) ?

We will see that under appropriate assumptions, indeed we can.

We already know there is something regarding convexity is involved there. In the case with differential constraints, Λ-convexity plays a role. In the case without differential constraints we saw if we start with solutions taking values in K, the limit of those solutions will land up taking values in the closed convex hull of K. Some **information on the derivatives enabled us to replace convexity by some more general and weaker notion of convexity**. We now ask if we start with solutions taking values in the **appropriate convex hulls of K, is it possible to land up, in the limit, in K** ?

Nonlinear PDEs as Differential Inclusions

We consider nonlinear PDEs that can be expressed as a system of linear PDEs (*balance laws*)

$$\sum_{i=1}^{m} A_i \partial_i z = 0 \tag{21}$$

coupled with a pointwise nonlinear constraint (*constitutive relations*)

$$z(x) \in K \subset \mathbb{R}^d \quad a.e \tag{22}$$

where $z : \Omega \subset \mathbb{R}^m \to \mathbb{R}^d$ is the unknown state variable.

Nonlinear PDEs as Differential Inclusions

We consider nonlinear PDEs that can be expressed as a system of linear PDEs (*balance laws*)

$$\sum_{i=1}^{m} A_i \partial_i z = 0 \tag{21}$$

coupled with a pointwise nonlinear constraint (*constitutive relations*)

$$z(x) \in K \subset \mathbb{R}^d \quad a.e \tag{22}$$

where $z : \Omega \subset \mathbb{R}^m \to \mathbb{R}^d$ is the unknown state variable.

In certain cases, this problem can be transformed into a differential inclusion problem.

If there exists a *potential* E and a matrix of partial differential operators $P(D)$ such that $z = P(D)E$ identically solves (21), that is,

$$\sum_{i=1}^{m} A_i \partial_i (P(D)E) = 0$$

then (21) and (22) together reduces to

$$P(D)E \in K \subset \mathbb{R}^d \quad a.e \,, \tag{23}$$

which is a *differential inclusion* problem, where E is usually a tensor and $P(D)$ is a matrix of partial differential operators, that is, each row of $P(D)$ is a partial differential operator.

Gradient Inclusion

We shall now illustrate the basic idea of solving a differential inclusion by using the most well-studied example of partial differential inclusions, namely the **gradient inclusion**. The presentation in the following section is influenced by works of Kirchheim and Sychev (Cf. [Sychev, 2001], [Sychev, 2006], [Kirchheim, 2003], [Kirchheim, 2001]).

Gradient Inclusion

We shall now illustrate the basic idea of solving a differential inclusion by using the most well-studied example of partial differential inclusions, namely the **gradient inclusion**. The presentation in the following section is influenced by works of Kirchheim and Sychev (Cf. [Sychev, 2001], [Sychev, 2006], [Kirchheim, 2003], [Kirchheim, 2001]).

Gradient inclusion

Consider the basic gradient inclusion problem with Dirichlet boundary data, where we will be looking for Lipschitz solutions,

$$\begin{aligned} \nabla u(x) &\in K \quad \text{for } a.e.\ x \quad \text{in} \quad \Omega \\ u(x) &= f(x) \quad \text{for all } x \quad \text{in } \partial\Omega \end{aligned} \tag{24}$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz, $K \subset \mathcal{M}^{m\times n}$ is closed and bounded.

In general, it is difficult to write down a solution at one strike. So we will be working in an auxiliary ‘working space’ and try to build the solution step by step.

In general, it is difficult to write down a solution at one strike. So we will be working in an auxiliary 'working space' and try to build the solution step by step.

We first choose a set $\mathcal{U} \in \mathcal{M}^{m\times n}$ of matrices such that

(a) we can realize the boundary data f with a "simple map" from Ω to $\mathbb{R}^m$ having gradients in $\mathcal{U}$ almost everywhere.

(b) "simple maps" with a gradient in $\mathcal{U}$ allow a gradual and local modification of their gradient distribution, which moves this distribution inside $\mathcal{U}$ towards K.

In general, it is difficult to write down a solution at one strike. So we will be working in an auxiliary 'working space' and try to build the solution step by step.

We first choose a set $\mathcal{U} \in \mathcal{M}^{m\times n}$ of matrices such that

(a) we can realize the boundary data f with a "simple map" from Ω to $\mathbb{R}^m$ having gradients in $\mathcal{U}$ almost everywhere.

(b) "simple maps" with a gradient in $\mathcal{U}$ allow a gradual and local modification of their gradient distribution, which moves this distribution inside $\mathcal{U}$ towards K.

The terminology "simple map" is a bit vague and it also slightly differs in (a) and (b). In (b), it usually means an affine map, whereas in (a), it usually means a piecewise affine map or almost everywhere locally affine Lipschitz map.
Once we are able to find such a set $\mathcal{U}$ and the approximate solutions, we can solve the differential inclusion problem by using convex integration or Baire Category arguments. Next we discuss these methods one by one and show how they are used to produce solution of the gradient inclusion problems.

Convex Integration

We start with the convex integration method, first introduced by Gromov in the far reaching generalization of Nash's work on embedding problems. Müller and Sverak adapted the method to the framework of Lipschitz solutions. A typical result in the spirit of convex integration is the following:

Theorem

Let S be a bounded subset of $\mathcal{M}^{m\times n}$ and let K be a compact subset of $\mathcal{M}^{m\times n}$. Assume that for every $A \in S$ and every $\epsilon > 0$ there is a piecewise affine function $\phi \in l_A + W_0^{1,\infty}(\Omega;\mathbb{R}^m)$ with the properties:

1) $D\phi \in (S \cup K)$ *a.e. in* Ω;

2) $\|dist(D\phi, K)\|_{L^1(\Omega)} \leqslant \epsilon\ |(\Omega)|$.

Then for each piecewise affine function $f \in W^{1,\infty}(\Omega;\mathbb{R}^m)$ with $Df \in S \cup K$ a.e. in Ω the problem (24) has a solution. Moreover, each ϵ-neighborhood of f in $L^\infty(\Omega;\mathbb{R}^m)$ norm contains a solution of this problem.

Here the notation $l_A + W_0^{1,\infty}(\Omega;\mathbb{R}^m)$ means the set of Lipschitz functions u such that $u = l_A$ on $\partial\Omega$ where l_A is a linear function with gradient equal to A.

Idea of the proof

Let f be a piecewise affine function with $Df \in (S \cup K)$ a.e. in Ω. The main point is to construct a sequence of piecewise affine functions $f_j : \Omega \to \mathbb{R}^m$ such that

$$Df_j \in (S \cup K) \text{ a.e. in } \Omega, \ \|dist(Df_j, K)\|_{L^1} \to 0, \tag{25}$$

$$f_j|_{\partial\Omega} = f|_{\partial\Omega}, \tag{26}$$

$$f_j \to f_\infty \text{ in } W^{1,1}(\Omega, \mathbb{R}^m). \tag{27}$$

This is done via explicit construction which relies heavily on **Vitali-type covering arguments** and **rescaling**.

Idea of the proof

Let f be a piecewise affine function with $Df \in (S \cup K)$ a.e. in Ω. The main point is to construct a sequence of piecewise affine functions $f_j : \Omega \to \mathbb{R}^m$ such that

$$Df_j \in (S \cup K) \text{ a.e. in } \Omega, \ \|dist(Df_j, K)\|_{L^1} \to 0, \tag{25}$$

$$f_j|_{\partial\Omega} = f|_{\partial\Omega}, \tag{26}$$

$$f_j \to f_\infty \text{ in } W^{1,1}(\Omega, \mathbb{R}^m). \tag{27}$$

This is done via explicit construction which relies heavily on **Vitali-type covering arguments** and **rescaling**.

The key point in the proof is that Controlled L^∞ convergence gives strong $W^{1,1}$ convergence.

We now introduce a definition needed to obtain a corollary of the above theorem.

Definition

A sequence of open sets $U_i \in \mathcal{M}^{m\times n}$ is an in-approximation of a set $K \in \mathcal{M}^{m\times n}$ if

(1) the U_i are uniformly bounded;

(2) $U_i \subset U_{i+1}^{lc}$;

(3) $U_i \to K$ in the following sense: if $F_i \in U_i$ and $F_i \to F$ then $F \in K$.

The subscript *lc* means lamination convex hull. Similarly rc-in-approximations can be defined using rank-1-convex hulls. A result similar to the corollary holds true in that case too.

Theorem

Suppose that $\{U_i\}$ *is an in-approximation of a compact set* $K \in \mathcal{M}^{m\times n}$ *and that* $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ *is* C^1*(or piecewise* C^1*) and satisfies* $f \in U_1$ *a.e. Then the problem (24) has a solution.*

Theorem

Suppose that $\{U_i\}$ is an in-approximation of a compact set $K \in \mathcal{M}^{m\times n}$ and that $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is C^1(or piecewise C^1) and satisfies $f \in U_1$ a.e. Then the problem (24) has a solution.

Theorem (14) and its rc-in-approximation version are actually both particular cases of Theorem (12) where the set S is taken as the union of the elements of the sets U_is. The proof of this fact relies on versions of the following lemma:

Lemma

Assume that $c \in \mathbb{R}^m$ and assume that $b \in \mathbb{R}^n$. Let $b_1 = t_1 b$, $b_2 = t_2 b$, where $t_2 < 0 < t_1$, and let $b_1, \dots, b_q$ be extreme points of a compact convex set with $0 \in int\ co\{b_1, \dots, b_q\}$. Define $B_i := c \otimes b_i$, $i \in \{1, \dots, q\}$. Then for each $\epsilon > 0$ there exists a piecewise affine function $\phi \in W_0^{1,\infty}(\Omega; \mathbb{R}^m)$ such that,

$$|\{x \in \Omega : D\phi(x) = B_1 \ or \ D\phi(x) = B_2\}| \geq |\Omega)| - \epsilon, \tag{28}$$

$$D\phi \in \{B_1, \dots, B_q\} \ a.e. \ in \ \Omega, \ \|\phi\|_{C(\Omega)} \leq \epsilon. \tag{29}$$

The proof relies on pyramidal construction and translation, covering and rescaling arguments.

Baire Category

The core of the Baire Category method lies in the fundamental observation that the gradient map $u \in (W^{1,\infty}, \|.\|_{L^\infty}) \to \nabla u \in L^p,\ p < \infty$ has a rather surprising continuity property in the framework of Baire Category.

Baire Category

The core of the Baire Category method lies in the fundamental observation that the gradient map $u \in (W^{1,\infty}, \|.\|_{L^\infty}) \to \nabla u \in L^p,\ p < \infty$ has a rather surprising continuity property in the framework of Baire Category.

Theorem

Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set and let $\mathcal{F} \subset Lip(\Omega; \mathbb{R}^m)$ equipped with the supremum norm $\|.\|_\infty$ be complete. Then for any $p < \infty$ the map $\nabla : f \to \nabla f$ is Baire class one maps from $(\mathcal{F}, \|.\|_\infty)$ into $L^p(\Omega; \mathcal{M}^{m\times n})$, i.e. is the pointwise limit of a sequence $\Delta_k : (\mathcal{F}, \|.\|_\infty) \to L^p(\Omega; \mathcal{M}^{m\times n})$ of continuous maps.
In particular, the typical f (i.e. all $f \in \mathcal{F}$ except those from a set of first Baire category in $\mathcal{F}$) is a point of continuity of the map ∇.

Definition

Given $\Omega \subset \mathbb{R}^n$ bounded and open, $\mathcal{U} \subset \mathcal{M}^{m \times n}$ bounded but not necessarily open, we put

$$\mathcal{P}(\Omega, \mathcal{U}) = \{u \in Lip(\Omega, \mathbb{R}^m) : u \text{ is piecewise affine and } \nabla u \in \mathcal{U} \text{ a.e. in } \Omega\}.$$

If there is in addition a map f from a superset of $\partial\Omega$ into $\mathbb{R}^m$ given, then we define the space

$$\mathcal{P}(\Omega, \mathcal{U}, f) = \{u \in \mathcal{P}(\Omega, \mathcal{U}) : u(x) = f(x) \text{ for all } x \in \partial\Omega\}.$$

Definition

Let $\mathcal{U}$, $K \subset \mathcal{M}^{m\times n}$ be given. We say that gradients in $\mathcal{U}$ are stable only near K if $\mathcal{U}$ is bounded, K is closed and for each $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that for all $A \in \mathcal{U}$ with $dist(A, K) > \epsilon$ there exists a piecewise affine $\phi \in Lip(\mathbb{R}^n, \mathbb{R}^m)$ with bounded support which satisfies

- $A + \nabla\phi(x) \in K_A$ for a.e $x \in \Omega$, where $K_A \subset \mathcal{U}$
- $\int |\nabla\phi(x)| > \delta\ |(supp\ \phi)|$

Theorem

Suppose that gradients in $\mathcal{U}$ are stable only near K. Given any $\Omega \subset \mathbb{R}^n$ bounded open set and $f : \Omega \to \mathbb{R}^m$ piecewise affine, the typical $u \in \overline{\mathcal{P}(\Omega, \mathcal{U}, f)}^{\infty}$ (in the sense of Baire category) is a solution of (24).

Here $\overline{\mathcal{P}(\Omega, \mathcal{U}, f)}^{\infty}$ means the closure of the space $\mathcal{P}(\Omega, \mathcal{U}, f)$ in L^{∞}-norm topology.

Incompressible Euler equation

Consider the incompressible Euler equation in n-space dimensions,

$$\left.\begin{aligned} \partial_t v + div(v \otimes v) + \nabla p = 0 \\ div\ \ v = 0 \end{aligned}\right\} \tag{30}$$

Incompressible Euler equation

Consider the incompressible Euler equation in n-space dimensions,

$$\left.\begin{aligned} \partial_t v + div(v \otimes v) + \nabla p = 0 \\ div \;\; v = 0 \end{aligned}\right\} \qquad (30)$$

Definition (Weak solutions)

A vectorfield $v \in L^2_{loc}(\mathbb{R}^n \times (0, T))$ is a weak solution of the incompressible Euler equations if

$$\int_0^T \int_{\mathbb{R}^n} (\partial_t \phi \cdot v + \nabla \phi : v \otimes v) \, \mathrm{d}x\mathrm{d}t = 0 \qquad (31)$$

for all $\phi \in C_c^\infty(\mathbb{R}^n \times (0, T); \mathbb{R}^n)$ with $div \;\; \phi = 0$ and

$$\int_0^T \int_{\mathbb{R}^n} v \cdot \nabla \psi \, \mathrm{d}x\mathrm{d}t = 0 \qquad (32)$$

for all $\psi \in C_c^\infty(\mathbb{R}^n \times (0, T))$.

Definition (Subsolutions)

Let $\overline{e} \in L^1_{loc}(\mathbb{R}^n \times (0, T))$ with $\overline{e} \geq 0$. A subsolution to the incompressible Euler equation with given kinetic energy density $\overline{e}$ is a triple

$$(v, \ u, \ q) : \mathbb{R}^n \times (0, T) \to \mathbb{R}^n \times \mathcal{S}_0^{n \times n} \times \mathbb{R}$$

with the following properties:

- $v \in L^2_{loc}$, $u \in L^1_{loc}$, q is a distribution;
- $$\begin{cases} \partial_t v + div \ u + \nabla q = 0 \\ div \ v = 0, \end{cases} \quad \text{in the sense of distributions;} \tag{33}$$
- $$v \otimes v - u \leq \frac{2}{n} \overline{e} I \quad \text{a.e..} \tag{34}$$

we define $U = \begin{pmatrix} u + qI_n & v \\ v & 0 \end{pmatrix}$ with $q = p + \frac{1}{n}|v|^2$, $u = v \otimes v - \frac{1}{n}|v|^2 I_n$. We also set $y = (x, t)$. Then (30) is equivalent to ,

$$div_y U = 0, \tag{35}$$

along with the set of constraint K defined by the relations between u and v and p and q.

we define $U = \begin{pmatrix} u + qI_n & v \\ v & 0 \end{pmatrix}$ with $q = p + \frac{1}{n}|v|^2$, $u = v \otimes v - \frac{1}{n}|v|^2 I_n$. We also set $y = (x, t)$. Then (30) is equivalent to ,

$$div_y U = 0, \tag{35}$$

along with the set of constraint K defined by the relations between u and v and p and q.

We denote by $\mathcal{M}$ the set of symmetric $(n+1) \times (n+1)$ matrices A such that $A_{(n+1)\times(n+1)} = 0$. Now, in view of the linear isomorphism

$$\mathbb{R}^n \times \mathcal{S}_0^n \times \mathbb{R} \ni (v, u, q) \mapsto U = \begin{pmatrix} u + qI_n & v \\ v & 0 \end{pmatrix} \in \mathcal{M}, \tag{36}$$

the wave cone corresponding to (35) can be written as

$$\Lambda = \left\{ (v, u, q) \in \mathbb{R}^n \times \mathcal{S}_0^n \times \mathbb{R} : det \begin{pmatrix} u + qI_n & v \\ v & 0 \end{pmatrix} = 0 \right\}. \tag{37}$$

Now we will present two lemmas which are crucial for the construction of the subsolutions. The first one is about a symmetry in the equation and the second one, in effect, reduces the problem to a differential inclusion problem.

Now we will present two lemmas which are crucial for the construction of the subsolutions. The first one is about a symmetry in the equation and the second one, in effect, reduces the problem to a differential inclusion problem.

Lemma (Galilean group symmetry)

Let $\mathcal{G}$ be the subgroup of $GL_{n+1}(\mathbb{R})$ defined by,

$$\{A \in \mathbb{R}^{(n+1)\times(n+1)} : detA \neq 0,\ Ae_{n+1} = e_{n+1}\}. \tag{38}$$

For every divergence free map $U : \mathbb{R}^{n+1} \to \mathcal{M}$ and every $A \in \mathcal{G}$ the map

$$V(y) = A^t \cdot U(A^{-t}y) \cdot A \tag{39}$$

is also a divergence free map $V : \mathbb{R}^{n+1} \to \mathcal{M}$

Proof.

It is easy to check that whenever $B \in \mathcal{M}$, then $A^t BA \in \mathcal{M}$ for all $A \in \mathcal{G}$. Now let $\phi \in C_c^\infty(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$ be a compactly supported test function and consider $\tilde{\phi}$ defined by

$$\tilde{\phi}(x) = A\phi(A^t x)$$

Then $\nabla\tilde{\phi}(x) = A\nabla\phi(A^t x)A^t$, and by a change of variables we obtain

$$\begin{aligned}
\int tr(V(y)\nabla\phi(y))\mathrm{d}y &= \int tr(A^t U(A^{-t}y)A\nabla\phi(y))\mathrm{d}y \\
&= \int tr(U(A^{-t}y)A\nabla\phi(y)A^t)\mathrm{d}y \\
&= \int tr(U(x)A\nabla\phi(A^t x)A^t)(detA)^{-1}\mathrm{d}x \\
&= (detA)^{-1}\int tr(U(x)\nabla\tilde{\phi}(x))\mathrm{d}x = 0,
\end{aligned}$$

since U is divergence free. But this implies that V is also divergence free. □

Lemma (Potential)

Let $E_{ij}^{kl} \in C^\infty(\mathbb{R}^{n+1})$ be functions for $i, j, k, l = 1, \ldots, n+1$ so that the tensor E is skew-symmetric in ij and kl, that is

$$E_{ij}^{kl} = -E_{ij}^{lk} = -E_{ji}^{kl} = E_{ji}^{lk} \ . \tag{40}$$

Then

$$U_{ij} = \mathcal{L}(E) = \frac{1}{2} \sum_{k,l} \partial_{kl}^2 (E_{il}^{kj} + E_{jl}^{ki}) \tag{41}$$

is symmetric and divergence free. If in addition

$$E_{(n+1)j}^{(n+1)i} = 0 \quad \textit{for every } i \textit{ and } j. \tag{42}$$

then U takes values in $\mathcal{M}$.

Proof.

Easy and direct calculation. □

Using these two lemmas, we can prove the following, which is essentially the core of the construction of De Lellis and Székelyhidi Jr. in [De Lellis and Székelyhidi, 2009]:

Using these two lemmas, we can prove the following, which is essentially the core of the construction of De Lellis and Székelyhidi Jr. in [De Lellis and Székelyhidi, 2009]:

Proposition (Localized plane waves)

Let $a = (v_0, u_0, q_0) \in \Lambda$ with $v_0 \neq 0$ and denote by σ the line segment joining the points a and $-a$. For every $\epsilon > 0$ there exists a smooth solution (v, u, q) of (35) (in view of the linear isomorphism (36)) with the properties:

- *the support of (v, u, q) is contained in $B_1(0) \subset \mathbb{R}^n_x \times \mathbb{R}_t$,*
- *the image of (v, u, q) is contained in the ϵ-neighborhood of σ,*
- $\int |v(x,t)| \mathrm{d}x\mathrm{d}t \geq \alpha |v_0|$,

where $\alpha > 0$ is a dimensional constant.

Proof.

Detailed proof can be found in [De Lellis and Székelyhidi, 2009]. We just sketch the argument emphasizing the crucial points. First note that using Lemma (22) and a standard covering/rescaling argument, it is enough to prove the proposition for the case where $\overline{U} \in \mathcal{M}$ is such that

$$\overline{U}e_1 = 0, \ \overline{U}e_{n+1} \neq 0. \tag{43}$$

Define

$$E^{i1}_{j1} = -E^{1i}_{j1} = -E^{i1}_{1j} = E^{1i}_{1j} = \overline{U}_{ij}\frac{\sin(Ny_1)}{N^2}. \tag{44}$$

and all the other entries equal to 0. Now choose a smooth cut-off function ϕ such that

- $|\phi| \leq 1$,
- $\phi = 1$ on $B_{\frac{1}{2}}(0)$,
- $supp(\phi) \subset B_1(0)$,

and consider the map

$$U = \mathcal{L}(\phi E). \tag{45}$$

Clearly, U is smooth and supported in $B_1(0)$. Using Lemma (23), U is $\mathcal{M}$-valued and divergence free. Also,

$$U(y) = \overline{U}\sin(Ny_1) \text{ for } y \in B_{\frac{1}{2}}(0),$$

and hence

$$\int |U(y)e_{n+1}|\mathrm{d}y \geq |\overline{U}e_{n+1}| \int_{B_{\frac{1}{2}(0)}} |\sin(Ny_1)|\mathrm{d}y \geq 2\alpha|\overline{U}e_{n+1}|, \tag{46}$$

for large enough N. Also note that

$$\|U - \phi\tilde{U}\|_\infty \leq \frac{C}{N}\|\phi\|_{C^2}, \tag{47}$$

where $\tilde{U} = \mathcal{L}(E)$. Hence by choosing N large enough, we can easily obtain $\|U - \phi\tilde{U}\|_\infty \leq \epsilon$. Since $|\phi| \leq 1$ and consequently the image of $\tilde{U}$ is contained in σ, this proves the proposition. □

Theorem

Let $\Omega \subset \mathbb{R}^n_x \times \mathbb{R}_t$ be a bounded open domain. There exist $(v, p) \in L^\infty(\mathbb{R}^n_x \times \mathbb{R}_t)$ solving the Euler equations

$$\begin{aligned} \partial_t v + div(v \otimes v) + \nabla p &= 0 \\ div\ v &= 0 \end{aligned}$$

such that

- $|v(x,t)| = 1$ *for a.e.* $(x,t) \in \Omega$
- $v(x,t) = 0$ *and* $p(x,t) = 0$ *for a.e.* $(x,t) \in \mathbb{R}^n_x \times \mathbb{R}_t \backslash \Omega$.

Instead of spelling out every details, which can be found in [De Lellis and Székelyhidi, 2009], we describe the geometric and functional analytic setup and outline the general strategy, specialized to the problem at hand.

The main idea is consider the sets

$$K = \{(v,u) \in \mathbb{R}^n \times \mathcal{S}_0^n : u = v \otimes v - \frac{1}{n}|v|^2 I_n,\ |v| = 1\}, \tag{48}$$

and

$$\mathcal{U} = int\ (conv\ K \times [-1,1]). \tag{49}$$

Clearly, a triple solving (35) (in the sense of the isomorphism) and taking values in the convex extreme points of $\overline{\mathcal{U}}$ is our desired solution. The plan is to prove $0 \in \mathcal{U}$, showing the existence of plane waves taking values in $\mathcal{U}$ and then add such plane waves to get an infinite sum

$$(v,u,q) = \sum_{i=1}^{\infty} (v_i, u_i, q_i) \tag{50}$$

with the properties that

- the partial sums $\sum_{i=1}^{\infty} (v_i, u_i, q_i)$ take values in $\mathcal{U}$,
- (v,u,q) is supported in Ω,
- (v,u,q) takes values in the convex extreme points of $\overline{\mathcal{U}}$ a.e. in Ω,
- (v,u,q) solves the linear partial differential equations (35)

The key functional analytic set up is the following:
Let

$$X_0 := \{(v,u,q) \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}_t) : (1),(2),(3) \text{ below hold } \} \tag{51}$$

(1) $supp(v,u,q) \subset \Omega$,

(2) (v,u,q) solves (35)(in the sense of the isomorphism) in $\mathbb{R}^n_x \times \mathbb{R}_t$

(3) $(v(x,t),u(x,t),q(x,t)) \in \mathcal{U}$ for all $(x,t) \in \mathbb{R}^n_x \times \mathbb{R}_t$.

We equip X_0 with the topology of L^∞-weak-$*$ convergence of (v,u,q) and we define X to be the closure of X_0 in this topology. Now the set X with this topology is a nonempty compact metrizable space. We fix a metric d^*_∞ inducing the weak-$*$ topology of L^∞ in X, so that (X, d^*_∞) is a complete metric space. Now, the identity map

$$I : (X, d^*_\infty) \to L^2(\mathbb{R}^n_x \times \mathbb{R}_t) \text{ defined by } (v,u,q) \mapsto (v,u,q)$$

is a Baire-1 map and therefore the set of points of continuity is residual in (X, d^*_∞).

Proving that $0 \in \mathcal{U}$ is done by considering the linear map

$$T : C(S^{n-1}) \to \mathbb{R}^n \times \mathcal{S}_0^n,\ \phi \mapsto \int_{S^{n-1}} \left(v, v \otimes v - \frac{I_n}{n} \right) \phi(v) \mathrm{d}\mu,$$

where μ is the Haar measure on S^{n-1}. Clearly, if

$$\phi \geq 0 \text{ and } \int_{S^{n-1}} \phi \mathrm{d}\mu = 1,$$

then $T(\phi) \in conv\ K$. Also note that $T(1) = 0 = T(\alpha)$ and using $\alpha = 1 - \int_{S^{n-1}} \psi \mathrm{d}\mu$ and choosing a ψ such that $\|\psi\|_{C(S^{n-1})} < \frac{1}{2}$, we obtain $T(B_{\frac{1}{2}}) \subset conv\ K$. Now we use intelligent choices of ϕ and the orthogonality in L^2 with a dimension counting argument to conclude T is surjective.

Using the fact that $0 \in \mathcal{U}$ and the construction of localized plane waves, we can show that

Lemma

There exists a dimensional constant $\beta > 0$ with the following property. Given $(v, u, q) \in X_0$ there exists a sequence $(v_k, u_k, q_k) \in X_0$ such that

$$\|v_k\|^2_{L^2(\Omega)} \geq \|v\|^2_{L^2(\Omega)} + \beta \left(|\Omega| - \|v\|^2_{L^2(\Omega)} \right)^2, \tag{52}$$

and

$$(v_k, u_k, q_k) \overset{*}{\rightharpoonup} (v, u, q) \ in \ L^\infty(\Omega). \tag{53}$$

Now if (v, u, q) is a point of continuity of the identity map I, passing to the limit in (52) we obtain $|v| = 1$ a.e in Ω, concluding the proof of Theorem (24). □

References I

Aubin, J.-P. and Cellina, A. (1984).
Differential inclusions, volume 264 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*.
Springer-Verlag, Berlin.
Set-valued maps and viability theory.

Cellina, A. (1980).
On the differential inclusion $x' \in [-1, +1]$.
Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8), 69(1-2):1–6 (1981).

Dacorogna, B. and Marcellini, P. (1997).
General existence theorems for Hamilton-Jacobi equations in the scalar and vectorial cases.
Acta Math., 178(1):1–37.

References II

Dacorogna, B. and Marcellini, P. (1999).
Implicit partial differential equations.
Progress in Nonlinear Differential Equations and their Applications, 37.
Birkhäuser Boston Inc., Boston, MA.

De Lellis, C. and Székelyhidi, J. L. (2009).
The Euler equations as a differential inclusion.
Ann. of Math. (2), 170(3):1417–1436.

Gromov, M. (1986).
Partial differential relations, volume 9 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*.
Springer-Verlag, Berlin.

Kirchheim, B. (2001).
Deformations with finitely many gradients and stability of quasiconvex hulls.
C. R. Acad. Sci. Paris Sér. I Math., 332(3):289–294.

References III

Kirchheim, B. (2003).
Rigidity and geometry of microstructures.
Habilitation thesis.

Sychev, M. A. (2001).
Comparing two methods of resolving homogeneous differential inclusions.
Calc. Var. Partial Differential Equations, 13(2):213–229.

Sychev, M. A. (2006).
A few remarks on differential inclusions.
Proc. Roy. Soc. Edinburgh Sect. A, 136(3):649–668.

Tartar, L. (1979).
Compensated compactness and applications to partial differential equations.
In *Nonlinear analysis and mechanics: Heriot-Watt Symposium, Vol. IV*, volume 39 of *Res. Notes in Math.*, pages 136–212. Pitman, Boston, Mass.

Thank you